ANA BEATRIZ VASCONCELOS PEREIRA

A NONSTANDARD DISSIPATIVE EFFECT FOR THE TIMOSHENKO SYSTEM

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Orientador: Prof. Dr. Rodrigo Nunes Monteiro

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“Wherever there is number, there is beauty.”
- Proclus

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ABSTRACT

In this study, we consider the Timoshenko system with coupled frictional dissipative effects through a real matrix B of order two. The objective is to study, using the Theory of Linear Semigroups, the existence and uniqueness of the solution for this system. Furthermore, by employing Prüss Theorem, we investigate the exponential stability of the Timoshenko system in question. We conclude that when the matrix B is a positive definite matrix, the system exhibits exponential decay. To complement the work, we present a particular case where the matrix B is not positive definite. However, exponential stability holds and is dependent on the equality of the wave speeds.

Keywords: Timoshenko system; Well-Posedeness; Exponential Stability.

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RESUMO

Neste trabalho consideramos o sistema Timoshenko com efeitos dissipativos friccionais acoplados por meio de uma matriz real B de ordem dois. O objetivo é estudar utilizando a teoria de semigrupos lineares, a existência e unicidade de solução deste sistema. Além disso, ao empregar o Teorema de Prüss, investigamos a estabilidade exponencial do sistema de Timoshenko em questão. Concluímos que quando a matriz B é uma matriz positiva definida, o sistema apresenta decaimento exponencial. Para complementar o trabalho, apresentamos um caso particular em que a matriz B não é positiva definida. No entanto, a estabilidade exponencial se mantém e depende da igualdade das velocidades das ondas.

Palavras-chave: Sistema de Timoshenko; Boa colocação; Estabilidade exponencial.

LISTA DE FIGURAS

SUMMARY

1 INTRODUCTION

The Timoshenko system is a system of partial differential equations which refers to a theoretical model used in the analysis of beams, developed by the Soviet engineer Stephen Timoshenko. Unlike the more simplified Euler-Bernoulli beam theory, the Timoshenko system considers the effects of shear deformation and rotation bending effects. This makes the Timoshenko system well-suited for analyzing structures such as thin beams or beams subjected to complex loading conditions.

The variables considered in the system are shear deformation and rotational bending effects, which are denoted by $\varphi = \varphi(x,t)$ and $\psi = \psi(x,t)$, both depending on the position $x \in [0,l]$, where l is the length of the beam, and the time $t \geq 0$, as it can be seen in Figure 1.1. In summary, the variables describe how the beam deforms in response to applied forces and moments.

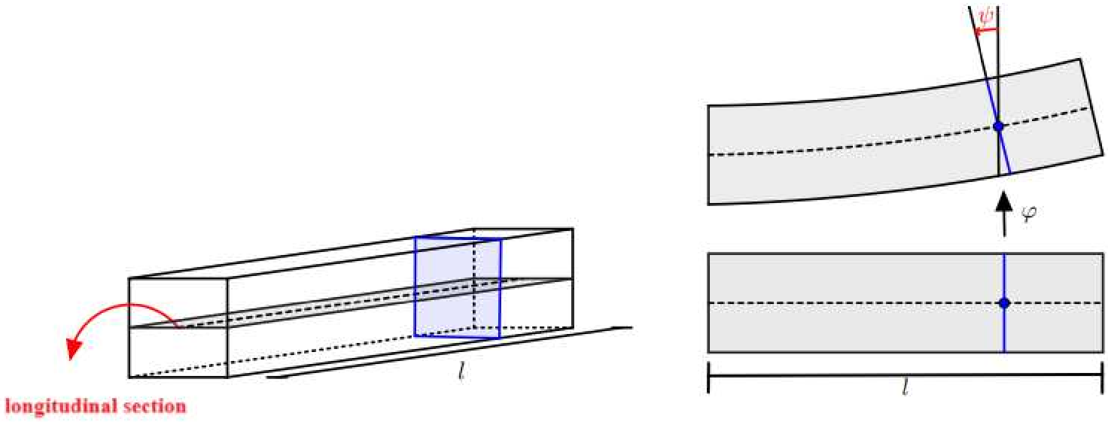

Figure 1.1: Timoshenko Beam. Font: Sozzo [14]

According to Timoshenko [15] and [16], the governing equations for φ and ψ are given by

$$\rho A \varphi_{tt} = S_x, \tag{1.1}$$

$$\rho I \psi_{tt} = M_x - S, \tag{1.2}$$

where ρ is the mass density, A is the area, I the moment of inertia of a cross section of a beam, S is the shear force and M the bending moment. Additionally, the elastic constitutive equations describing the relationship for shear force and bending moment are expressed as follows

$$S = k'GA(\varphi_x + \psi), \tag{1.3}$$

$$M = EI\psi_x, \tag{1.4}$$

where k' is a shear correction factor, and G and E denote the shear and Young's modulus, respectively. The later, illustratively, are represented in Figure 1.2 where the arrows depict the applied force on the object.

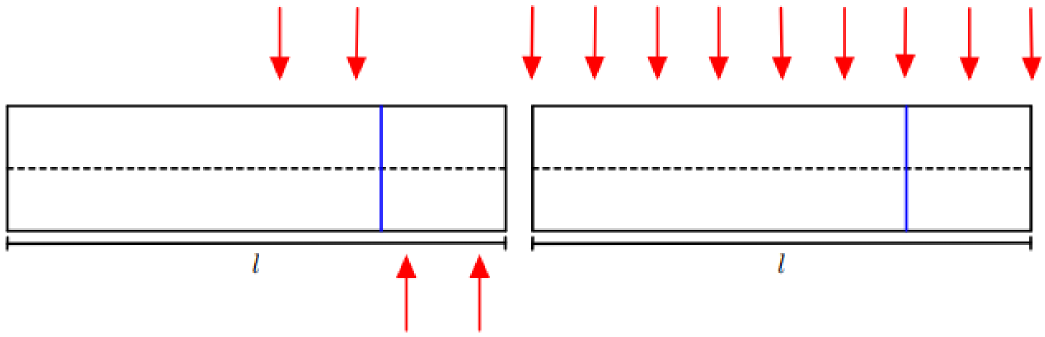

Figure 1.2: Shear force and Bending moment. Font:[14]

Replacing (1.3)-(1.4) into (1.1)-(1.2) and denoting the constants as

$$\rho_1 = \rho A,\ \rho_2 = \rho I,\ k = k'GA,\ b = EI,$$

we obtain the following Timoshenko system

$$\begin{aligned} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) &= 0. \end{aligned}$$

The focal point of our exploration lies within the mathematical framework of the Timoshenko problem, with dissipative effect motivated by Alves [3], captured by the following system of partial differential equations

$$\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + b_{11}\varphi_t + b_{12}\psi_t = 0 \ \text{ in } \ (0, l) \times (0, \infty), \tag{1.5}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + b_{21}\varphi_t + b_{22}\psi_t = 0 \ \text{ in } \ (0, l) \times (0, \infty). \tag{1.6}$$

In this system, $b_{11},\ b_{12},\ b_{21},\ b_{22} \in \mathbb{R}$ constitute the damping matrix B. We further assume that the beam is fixed at the ends $0, l$. The crux of our investigation lies in demonstrating the exponential stability of the system above, under the proper assumptions on $b_{11}, b_{12}, b_{21}, b_{22}$.

Before delving further into the purpose of this work, let us mention some results about exponential stability. Raposo et al. [12] investigated the Timoshenko system with frictional dissipation acting on both equations. Precisely, the system (1.5)-(1.6) with $b_{11} = b_{22} = 1$ and $b_{12} = b_{21} = 0$, as given by

$$\begin{aligned} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \varphi_t &= 0 \ \text{ in } \ (0, l) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \psi_t &= 0 \ \text{ in } \ (0, l) \times (0, \infty). \end{aligned}$$

The main result in [12] is that the problem exhibits exponential stability.

Rivera et al. [2] explored the Timoshenko systems with frictional dissipation affecting only the vertical displacement, specifically the system (1.5)-(1.6) with $b_{11} = 1$ and $b_{12} = b_{21} = b_{22} = 0$, as it follows

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi) + \varphi_t = 0 \ \text{ in } \ (0,l)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0 \ \text{ in } \ (0,l)\times(0,\infty). \end{aligned}$$

In this setup a condition for the exponential stability is $\frac{k}{b} = \frac{\rho_1}{\rho_2}$, in other words, the wave speeds are the same.

In Soufyane [13], the damping matrix coefficients were defined by $b_{22} = \beta(x)$ and $b_{11} = b_{12} = b_{21} = 0$, where $\beta > 0$ is a continuous function of the spatial variable. The system is given by

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi) = 0 \ \text{ in } \ (0,l)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta(x)\psi_t = 0 \ \text{ in } \ (0,l)\times(0,\infty). \end{aligned}$$

This reference demonstrates exponential stability under the condition of equal wave speeds.

In light of the referenced works, our aim is to extend the findings derived by Raposo et al. [12] by exploring a wider range of matrices. To achieve these objectives, this work will be structured as follows: In Chapter 2, we will establish the theoretical groundwork for the present work, presenting some results in functional analysis, Sobolev spaces, and semigroups of linear operators. We will provide references where the statements and proofs of these results can be found, among which the following stand out: the Theorems of Lumer-Phillips (Theorem 2.45) and Lax-Milgram (Theorem 2.14), which will guarantee the existence and uniqueness of a solution for our system. Furthermore, we will present the Prüss Theorem (Theorem 2.47), which will be used to obtain the exponential stability results. In Chapter 3, we will use the previous results to study the well-posedness of our system. This chapter is divided into presenting the problem, formulating the semigroup, and finally, proving its well-posedness. In Chapter 4, we will show that the system is exponentially stable, considering the damping matrix B as a positive definite matrix. Finally, in Chapter 5, we will work on a similar model to [13] with constant damping parameter $\beta = 1$ where the hypothesis of B being a positive definite matrix is not satisfied, but it stability holds under proper assumptions.

2 PRELIMINARIES

This chapter aims to provide a theoretical foundation for this work. The presented results are preliminary and do not encompass the entire conceptual framework, highlighting only the main results used.

2.1 FUNCTIONAL ANALYSIS

In this section we will denote $\mathbb{K}$ as the field of real number $\mathbb{R}$ or the field of complex number $\mathbb{C}$.

Lemma 2.1 (Young Inequality I). *Let a and b be non-negative constants and $1 < p, q < \infty$, such that, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. See Evans [6], page 622, Section B.2. □

Note 1. The numbers p and q which satisfies $\frac{1}{p} + \frac{1}{q} = 1$ are known as the conjugate exponents.

Lemma 2.2 (Young Inequality II). *Let a and b non-negatives constants, $1 < p, q < \infty$ conjugate exponents. Then, given $\epsilon > 0$ it holds that*

$$ab \leq \epsilon a^p + C_\epsilon b^q,$$

where $C_\epsilon = \frac{(\epsilon p)^{-\frac{q}{p}}}{q}$.

Proof. See page 622 from [6], Section B.2. □

Definition 2.3 (Norm). *Let X be a vector space over $\mathbb{K}$. A norm in X is a function $\| \cdot \|_X : X \longrightarrow \mathbb{R}$ with the following properties*

(N1) $\|x\|_X \geq 0,$

(N2) $\|x\|_X = 0 \Leftrightarrow x = 0,$

(N3) $\|\alpha x\|_X = |\alpha| \, \|x\|_X, \forall x \in X$ *and* $\alpha \in \mathbb{K},$

(N4) $\|x + y\|_X \leq \|x\|_X + \|y\|_X, \forall \, x, \, y \in X.$

Definition 2.4 (Normed space). *A normed space is a vector space with a well defined norm.*

Definition 2.5. *A sequence in set X is a mapping $x : \mathbb{N} \longrightarrow X$ which we denote by $x_n \equiv x(n)$ and $x(\mathbb{N}) \equiv (x_n)_{n \in \mathbb{N}}$. A subsequence of $(x_n)_{n\in\mathbb{N}}$ is a restriction $x|_{\mathbb{N}'} : \mathbb{N}' \longrightarrow X$ of the x function to a infinity subset $\mathbb{N}' \subset \mathbb{N}$.*

Definition 2.6. *Consider* $(x_n)_{n\in\mathbb{N}}$ *a sequence in a normed vector space* $(X, \|\cdot\|_X)$. *The sequence is said to be*

(i) *bounded if there is* $M > 0$, *such that,* $\|x_n\|_X \leq M, \ \ \forall n \in \mathbb{N}$;

(ii) *convergent in* X *if exists* $x \in X$ *satisfying*

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \textit{ such that, } \ \|x_n - x\|_X < \epsilon, \forall n > n_0,$$

(iii) *a Cauchy sequence in* X *if*

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \textit{ such that, } \ \|x_m - x_n\|_X < \epsilon, \forall m, n > n_0.$$

Definition 2.7 (Banach Space). *A normed vector space* $(X, \|\cdot\|_X)$ *is called a Banach space if every Cauchy sequence in* X *converges in* X.

Note 2. We will denote $\mathcal{L}(X, Y)$ the set of linear and bounded operators $A : X \longrightarrow Y$. If $X = Y$, we denote by $\mathcal{L}(X)$.

Theorem 2.8. *Consider* X *and* Y *two vector normed spaces and* $A : X \longrightarrow Y$ *a linear operator. Then,* A *is bounded if, and only if,* A *is continuous.*

Proof. See Kreyszig [7] Theorem 2.7-9 (a), page 97. □

Definition 2.9 (Inner product). *Let* X *be a vector space over* $\mathbb{K}$. *An inner product is a function* $(\cdot,\cdot)_X : X \times X \longrightarrow \mathbb{K}$ *with the following properties*

(P1) *For all* $x \in X$, $(x, x)_X \geq 0$ *and* $(u, u)_X = 0 \Leftrightarrow x = 0$,

(P2) *For all* $x, y \in X$, *holds* $(x, y)_X = (y, x)_X$,

(P3) *For all,* $x, y \in X$, *and* $\alpha, \beta \in \mathbb{K}$ *holds* $(\alpha x, y)_X = \alpha(x, y)_X$ *and* $(x, \beta y)_X = \overline{\beta}(x, y)_X$,

(P4) *For all* $x, y, z \in X$, $(x + y, z)_X = (x + z)_X + (x + y)_X$.

Definition 2.10. *Let* X *be a vector space equipped with inner product* $(\cdot,\cdot)_X$. *The norm defined by* $\|x\|_X = \sqrt{(x, x)_X}$ *is said to be induced by the inner product* $(\cdot,\cdot)_X$.

Definition 2.11 (Hilbert space). *A Banach space* $(X, \|\cdot\|_X)$ *is called a Hilbert space when* X *is complete with respect to the norm induced by the inner product on* X.

Definition 2.12. *Let* X *and* Y *be* $\mathbb{K}$ *vector spaces. A sesquilinear form is a function in two variables* $a : X \times Y \longrightarrow \mathbb{K}$ *that satisfies the following properties*

(i) $a(x + y, z) = a(x, z) + a(y, z), \ \ \forall\, x, y \in X \ \textit{ and } \ z \in Y$;

(ii) $a(x, y + z) = a(x, y) + a(x, z), \ \ \forall\, x \in X \ \textit{ and } \ y, z \in Y$;

(iii) $a(\lambda x, y) = \lambda a(x, y), \ \ \forall\, x \in X, y \in Y$ *and* $\lambda \in \mathbb{K}$;

(iv) $a(x, \lambda y) = \overline{\lambda} a(x, y), \ \ \forall\, x \in X, y \in Y$ *and* $\lambda \in \mathbb{K}$.

Definition 2.13. *Let X and Y normed vector spaces. A sesquilinear form $a : X \times Y \longrightarrow \mathbb{K}$ is*

(i) *continuous if there is $C > 0$, such that, $|a(x, y)| \leq C\|x\|_X\|y\|_Y$, for all $(x, y) \in X \times Y$;*

(ii) *coercive if there is $C > 0$, such that, $\mathrm{Re}\{a(x, x)\} \geq C\|x\|_X^2$, for all $x \in X$.*

Theorem 2.14 (Lax-Milgram). *Consider X a real (complex) Hilbert space and a continuous and coercive bilinear (sesquilinear) form $a : X \times X \longrightarrow \mathbb{K}$. Then, for every bounded linear (antilinear) functional h, there is a unique $x \in X$, such that, $a(x, y) = h(y)$, for all $y \in X$.*

Proof. For the real case, see Brezis [4], Corollary 5.8, page 140. For the complex case, see Oden [9], Corollary 6.6.2, page 595. □

Definition 2.15. *A normed space X is called reflexive if the canonical map*

$$\begin{aligned} J : X &\rightarrow X'' \\ x &\mapsto g_x \end{aligned}$$

is surjective, where $g_x : X' \longrightarrow \mathbb{K}$ is given by $g_x(f) = f(x)$.

Theorem 2.16. *Every Hilbert space is reflexive.*

Proof. See [7], Theorem 4.6-6, page 242. □

Definition 2.17. *Let X be a Hilbert space. We say that operator $A : D(A) \subset X \longrightarrow X$ is dissipative if* $\mathrm{Re}\{(Ax, x)_X\} \leq 0, \ \forall x \in D(A)$.

Theorem 2.18. *Let $A : D(A) \subset X \longrightarrow X$ be a dissipative operator, such that, the operator $I_X - A$ is surjective. If X is a reflexive space, then $\overline{D(A)} = X$.*

Proof. See Pazy [10], Theorem 4.6, page 16. □

Definition 2.19. *Let X and Y be Banach spaces. A linear operator $A : X \longrightarrow Y$ is said to be compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ there is a subsequence $(x_{n_j})_{j\in\mathbb{N}}$, such that, $(A(x_{n_j}))_{j\in\mathbb{N}}$ converges in Y.*

Definition 2.20. *Consider the linear operator A of a Banach space X. The set formed by $\lambda \in \mathbb{C}$ for which the linear operator $(\lambda I_X - A)$ is invertible, its inverse is bounded and densely defined, is said to be the* ***resolvent set*** *of T and denoted by $\rho(A)$.*
The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is the ***spectrum*** *of T. For $\lambda \in \rho(A)$, we denote by $R(\lambda, A) = (\lambda I_X - A)^{-1}$ the* ***resolvent*** *of A.*

Definition 2.21. *Consider X a Banach space and $A : D(A) \subset X \longrightarrow X$ a linear operator. We say that operator A has a compact resolvent if exists $\lambda \in \rho(A)$, such that, $(\lambda I_X - A)^{-1}$ is compact.*

Proposition 2.1. *Let $(X, \|\cdot\|_X)$ be a Banach space and $A : D(A) \subset X \longrightarrow X$ a linear operator with $\rho(A) \neq \emptyset$. Then, operator A has a compact resolvent if, and only if, the inclusion map $i : (D(A), \|\cdot\|_{D(A)}) \longrightarrow (X, \|\cdot\|_X)$ is compact.*

Proof. See Engel [5] Proposition 5.8, page 107. □

Proposition 2.2. *Consider a Banach space X and a linear operator $A : D(A) \subset X \longrightarrow X$ with compact resolvent. Then, $\sigma(A)$ is composed only by eigenvalues of A.*

Proof. See [5] Corollary 1.15, Page 162. □

2.2 THE SPACE L^p

Definition 2.22 **($L^p(\Omega)$ spaces).** *Let $\Omega \subset \mathbb{R}^n$ open and $0 < p < \infty$. Consider $\mathcal{L}^p(\Omega)$ the set of all measurable functions $f : \Omega \longrightarrow \mathbb{C}$, such that, $|f|^p$ is integrable in the Lebesgue sense in Ω, that is*

$$\mathcal{L}^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{C} \;\middle|\; f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

The functions $f, g \in \mathcal{L}^p$ are said to be equivalents ($f \sim g$), if $f = g$ almost everywhere in Ω. We will indicate by $L^p(\Omega)$ the set

$$L^p(\Omega) = \mathcal{L}^p \backslash \sim .$$

The $L^p(\Omega)$ norm is given by

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

For $p = \infty$, we define

$$L^\infty(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \;\middle|\; f \text{ is bounded almost everywhere in } \Omega \right\}.$$

Definition 2.23. *Consider the function $f : \Omega \longrightarrow \mathbb{C}$. We call the essential supremum of f in Ω the number*

$$\sup_{x\in\Omega} ess|f(x)| = \inf\left\{ K \;\middle|\; |f(x)| \leq K \text{ almost everywhere in } \Omega \right\}.$$

The norm on $L^\infty(\infty)$ by

$$\|f\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} ess|f(x)| = \inf\left\{ C > 0 \;\middle|\; |f(x)| \leq C \text{ almost everywhere in } \Omega \right\}.$$

Theorem 2.24. *If* $1 \leq p \leq \infty$ *then* $L^p(\Omega)$ *is a Banach space.*

Proof. See [4] Theorem 4.8, pages 93 and 94. □

Theorem 2.25 (Hölder Inequality). *Let* $\Omega \subset \mathbb{R}^n$ *open and* p, q *conjugate exponents where* $1 \leq p \leq \infty$*. If* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$*, then* $fg \in L^1(\Omega)$ *and*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Proof. See [4] Theorem 4.6, page 92. □

Corollary 2.3. *Let* $1 \leq p \leq q \leq \infty$*. If* $f \in L^q(\Omega)$ *and* $|\Omega| < \infty$*, then* $f \in L^p(\Omega)$ *and*

$$\|f\|_{L^p(\Omega)} \leq |I|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(\Omega)}.$$

Proof. See Adams and Fournier [1] Theorem 2.14, page 28. □

Theorem 2.26 (Minkowiski Inequality). *Consider* $f, g \in L^p(\Omega)$ *and* $1 \leq p \leq \infty$*. Then,*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

Proof. See [4] Theorem 4.7, pages 93. □

Theorem 2.27. *The space* $L^2(\Omega)$ *is a Hilbert space with the following inner product*

$$(u, v)_2 = \int_\Omega u(x)\overline{v}(x)\, dx, \ \forall u, v \in L^2(\Omega).$$

Proof. See [1] Corollary 2.18, page 31. □

Note 3. Note that, the inner product in $L^2(\Omega)$ induces the following a norm

$$\|u\|_2^2 = (u, u)_2 = \int_\Omega |u(x)|^2 \, dx, \, \forall u \in L^2(\Omega).$$

Proof. See [4], page 93, Theorem 4.7. □

Definition 2.28. *Let* $\Omega \subset \mathbb{R}^n$ *be a open set and* $u : \Omega \longrightarrow \mathbb{C}$ *a continuous mapping. The support of* u *is*

$$\text{supp}(u) = \overline{\{x \in \Omega \mid u(x) \neq 0\}}^{\Omega}.$$

We will denote by $C_0(\Omega) = \{u \in C(\Omega) \mid \text{supp}(u) \text{ is compact}\}$.

Definition 2.29. *The space* $C_0^\infty(\Omega)$ *is defined by*

$$C_0^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{K} \mid \textit{infinitely differentiable with compact support}\}.$$

Proposition 2.4. *Let* $\Omega \subset \mathbb{R}^n$ *an open set and* $1 \leq p < \infty$, *then* $C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega)$.

Proof. See [4] Corollary 4.23, page 109. □

2.3 ONE-DIMENSIONAL SOBOLEV SPACES

2.3.1 The space $W^{1,p}(I)$

Let $I = (a, b)$ be an interval with $-\infty \leq a < b \leq +\infty$ and $1 \leq p \leq \infty$.

Definition 2.30. *The Sobolev space $W^{1,p}(I)$ is defined as*

$$W^{1,p}(I) = \left\{ u \in L^p(I) \mid \exists g \in L^p(I) \text{ where } \int_I u\varphi' \, dx = -\int_I g\varphi \, dx, \ \forall \varphi \in C_0^1(I) \right\}.$$

Particularly, when $p = 2$, is denoted as $W^{1,2}(I) = H^1(I)$, that is

$$H^1(I) = \left\{ u \in L^2(I) \mid \exists g \in L^2(I) \text{ where } \int_I u\varphi' \, dx = -\int_I g\varphi \, dx, \ \forall \varphi \in C_0^1(I) \right\}.$$

Note 4. Given $u \in W^{1,p}(I)$, the function g is called as weak derivative of u in $W_{1,p}(I)$ and will be denoted by u'. The weak derivative, conditioned on its existence, is unique except on a set with Lebesgue measure zero.

Note 5. The space $W^{1,p}(I)$ is equipped with the norm

$$\|u\|_{W^{1,p}(I)} = \|u\|_{L^p(I)} + \|u'\|_{L^p(I)}$$

or with the equivalent norm

$$\|u\|_{W^{1,p}(I)} = (\|u\|_{L^p(I)}^p + \|u'\|_{L^p(I)}^p)^{\frac{1}{p}}.$$

Note 6. When $p = 2$, the space $W^{1,2}(I) = H^1(I)$ is a Hilbert space with the inner product defined by

$$(u, v)_{H^1(I)} = (u, v)_{L^2(I)} + (u', v')_{L^2(I)}.$$

Theorem 2.31. *The Sobolev space $W^{1,p}(I)$ is a Banach space for $1 \leq p \leq \infty$.*

Proof. See [4] Proposition 8.1, page 203. □

Lemma 2.32. *Consider $u, v \in W^{1,p}(I)$ where $1 \leq p < \infty$. Then,*

(i) *$uv \in W^{1,p}(I)$ and $(uv)' = u'v + uv'$.*

(ii) *Also, the following formula holds*

$$\int_a^b u'(s)v(s)ds = u(b)v(b) - u(a)b(a) - \int_a^b u(s)v'(s)ds.$$

Proof. See [4] Corollary 8.10, page 215. □

2.3.2 The space $W_0^{1,p}(I)$

Definition 2.33. *Let $1 \leq p \leq \infty$. The space $W_0^{1,p}(I)$ is defined as*

$$W_0^{1,p}(I) = \overline{C_0^1(I)}^{W^{1,p}(I)}.$$

If $p = 2$, then

$$W_0^{1,2}(I) = H_0^1(I) = \overline{C_0^1(I)}^{H^1(I)}.$$

Note 7. The space $W_0^{1,p}(I)$, $1 \leq p < \infty$ are normed vector spaces equipped with $W^{1,p}(I)$ norm.

Theorem 2.34. *Consider $u \in W^{1,p}(I)$. Then, $u \in W_0^{1,p}(I)$ if, and only if, $u = 0$ in ∂I.*

Proof. See [6] Theorem 2 , page 259. □

Note 8. We have that $W_0^{1,p}(I) = \{u \in W^{1,p}(I); u = 0 \text{ in } \partial I\}$.

Theorem 2.35 (Poincaré Inequality). *If I is a bounded interval. Then, exists a constant $c_p = c_p(med(I)) > 0$ such that*

$$\|u\|_{W^{1,p}}(I) \leq c_p \|u'\|_{L^p(I)}, \ \forall u \in W_0^{1,p}(I).$$

Proof. See [4] Proposition 8.13, page 218. □

Note 9. If I is bounded, $W_0^{1,p}(I)$ is a normed vector space with the following norm

$$\|u\|_{W_0^{1,p}(I)} = \|u'\|_{L^p(I)}.$$

Note 10. If $p = 2$ then $H_0^1(I) = W_0^{1,2}(I)$ is a Hilbert space with the inner product is given by

$$(u, v)_{H_0^1(I)} = \int_I v'(x)u'(x)\, dx.$$

This inner product induces the following norm

$$\|u\|_{H_0^1(I)}^2 = (u, u)_{H_0^1(I)} = \int_I (u'(x))^2\, dx = \|u'\|_{L^2(I)}^2,$$

equivalent to the norm $\|u\|_{W_0^{1,p}(I)} = \|u'\|_{L^p(I)}$.

2.3.3 The space $W^{m,p}(I)$

Definition 2.36. *Let $1 \leq p \leq \infty$, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$, $|\alpha| = \alpha_1 + \ldots \alpha_N$ and $u, g \in L^p(I)$. We say g is the $alpha - th$ order weak derivative of u when*

$$\int_I u D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_I g\varphi\, dx, \ \forall \varphi \in C_0^\infty(I).$$

Definition 2.37. *Consider* $1 \le p \le \infty$ *and* $m \in \mathbb{Z}_+$. *We define the Sobolev space* $W^{m,p}(I)$ *as the vector subspace of* $L^p(I)$ *given by*

$$W^{m,p}(I) = \{u \in L^p(I) \mid \exists\, u',\, u'', \dots,\, u^{(m)} \in L^p(I)\},$$

where $u', u'', \dots, u^{(m)}$ *denote the weak derivatives of order* $1, 2, \dots, m$, *respectively. When* $p = 2$ *we use the notation* $W^{m,2}(I) = H^m(I)$.

Note 11. Here, the first and second order weak derivatives will be denoted by u_x and u_{xx}, respectively.

Theorem 2.38. *The space* $W^{m,p}(I)$, $1 \le p \le \infty$ *is a Banach space with the following norm*

$$\|u\|_{W^{m,p}(I)} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|^p_{L^p(I)} \right)^{\frac{1}{p}}, \;\; 1 \le p < \infty,$$

$$\|u\|_{W^{m,\infty}(I)} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^\infty(I)}, \;\; p = \infty.$$

Proof. See [6] Theorem 2, page 249. □

Note 12. The spaces $W^{m,2}(I) = H^m(I)$ are Hilbert spaces with the following inner product

$$(u, v)_{H^m(I)} = \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v)_{L^2(I)}, \;\; \forall u, v \in H^m(I).$$

Definition 2.39. *Consider* $1 \le p < \infty$. *The space* $W_0^{m,p}(I)$ *is defined as*

$$W_0^{m,p}(I) = \overline{C_0^m(I)}^{W^{m,p}}.$$

When $p = 2$ *it is also denoted*

$$W_0^{m,2}(I) = \overline{C_0^m(I)}^{W^{m,2}(I)} = H_0^m(I).$$

Definition 2.40. *Let* X *and* Y *Banach spaces with* $Y \subset X$. *We say that* Y *is continuously embedded in* X *if inclusion map*

$$\begin{aligned} i : Y &\to X \\ y &\mapsto i(y) = y, \end{aligned}$$

is continuous. In this context, we write $Y \hookrightarrow X$. *We say* Y *is compactly embedded into* X *when the inclusion map* $i : Y \longrightarrow X$ *is compact. We denote the compact embedding of* Y *into* X *by* $Y \overset{c}{\hookrightarrow} X$.

Theorem 2.41 (Sobolev Embeddings). *We have that*

(i) $W^{m,p}(I) \subset L^{\infty}(I)$, *for all* $1 \leq p \leq \infty$ *and every* $m \geq 1$, *with a continuous embedding.*

(ii) *If* I *is bounded then* $W^{m,p}(I) \subset C^{m-1}(\overline{I})$ *for all* $1 < p \leq \infty$ *and every* $m \geq 1$ *with a continuous and compact embedding.*

(iii) *If* I *is bounded the* $W^{m,p}(I) \subset L^{q}(I)$ *for all* $1 \leq q < \infty$ *and every* $m \geq 1$, *where* $\frac{1}{p} + \frac{1}{q} = 1$, *with a compact embedding.*

Proof. See [4] Theorem 8.8, page 212. □

2.4 SEMIGROUPS OF LINEAR OPERATORS

Throughout this section we will introduce some results and definitions of the semigroups of linear operators theory.

Definition 2.42. *Let* X *be a Banach space. A one parameter family* $\{S(t)\}_{t\geq 0}$, *of bounded linear operators from* X *into* X *is a semigroup of bounded linear operator on* X *if*

(i) $S(0) = I_X$*;*

(ii) $S(t+s) = S(t)S(s)$, *for every* $t, s \geq 0$.

(iii) *Moreover, the semigroup is called as a* C_0*-semigroup, if* $\lim_{t\to 0^+} \|S(t) - x\|_X = 0$, *for each* $x \in X$.

Definition 2.43. *The linear operator* $A : D(A) \subset X \to X$ *defined as*

$$D(A) = \left\{ x \in X \;\middle|\; \lim_{t\to 0} \frac{S(t)(x) - I_X(x)}{t} \;\; \text{exists} \right\}$$

and

$$A(x) = \lim_{t\to 0} \frac{S(t)(x) - I_X(x)}{t}, \;\; \forall x \in D(A)$$

is the infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$.

Also, note that the domain $D(A)$ *of the operator* A *can be rewritten as*

$$D(A) = \{x \in X \mid Ax \in X\}.$$

Theorem 2.44. *Consider the abstract Cauchy problem*

$$\begin{aligned} &u'(t) = A(u(t)), \;\; t > 0, \\ &u(0) = u_0. \end{aligned} \tag{2.1}$$

If operator A is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ in a Banach Space X, then for each $u_0 \in D(A)$, there is a unique solution of (2.1) *in*

$$u \in C([0,\infty);\ D(A)) \cap C^1((0,\infty); X).$$

Proof. See [4] Theorem 7.4, page 185. □

Theorem 2.45 (Lumer-Phillips). *If A is a infinitesimal generator of a contraction C_0-semigroup in a Banach space X, then*

(i) *A is a dissipative operator;*

(ii) $\text{Im}(\lambda I_X - A) = X$, for all $\lambda > 0$.

Reciprocally, if

(iii) *$D(A)$ is dense in X;*

(iv) *A is a dissipative operator;*

(v) $\text{Im}(\lambda_0 I_X - A) = X$ *for some* $\lambda_0 > 0$.

Then A is the infinitesimal generator of a contraction C_0-semigroup in X.

Proof. See [10] Theorem 4.3, page 14. □

Definition 2.46. *A semigroup $\{S(t)\}_{t\geq 0}$ on X is exponentially stable if there are positive constants C and k, such that*

$$\|S(t)u_0\|_X \leq Ce^{-kt}\|u_0\|_X,\ \forall t \geq 0 \text{ and } u_0 \in X.$$

Theorem 2.47 (Prüss). *Let $\{S(t)\}_{t\geq 0}$ be a contraction semigroup on a Hilbert Space X. Then, $\{S(t)\}_{t\geq 0}$ is exponentially stable if, and only if, the following conditions holds*

$$\rho(A) \supseteq \{i\lambda | \lambda \in \mathbb{R}\} = i\mathbb{R} \text{ and } \limsup_{|\lambda|\to\infty} \|(i\lambda I_X - A)\|^{-1}_{\mathcal{L}(X)} < \infty.$$

Proof. See Prüs [11], Theorem 4 and Corollary 5, page 853-855 . □

2.5 Positive Definite Matrix

Definition 2.48 (Hermitian Matrix). *A square matrix B is said to be Hermitian (or symmetric if $\mathbb{K} = \mathbb{R}$) if $B = \overline{B}^{\top}$, where superscript $\top$ denotes the vector transpose operation.*

Definition 2.49 (Positive Definite Matrix). *A square Hermitian matrix B of order $n \times n$ is said to be a positive definite if*

$$\overline{x}^{\top} \cdot Bx > 0, \ \forall x \in \mathbb{C}^n - \{0\},$$

where $\cdot$ denotes the inner product in $\mathbb{R}^n$.

Theorem 2.50. *If $B = [b_{ij}]$ is a positive definite matrix, then*

(i) $b_{ii} > 0$*;*

(ii) $\det(B) > 0$.

Proof. **(i)** Since B is a positive definite matrix, then for all $x \in C^n - \{0\}$, we have $\overline{x}^{\top} \cdot Bx > 0$. In particular, consider $x = e_i$, where e_i is the standard basis vector. Then,

$$\overline{e_i}^{\top} \cdot Be_i > 0.$$

However, $\overline{e_i}^{\top} \cdot Be_i = b_{ii}$. Hence $b_{ii} > 0$ for $i = 1, \cdots, n$.

(ii) See Leon [8], page 339, Property II.

□

Lemma 2.51. *Let $b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{R}$ and B be a symmetric matrix defined by*

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix},$$

where $b_{11}, b_{22} > 0$. Then there is a positive constant C, such that,

$$\frac{\det(B)}{b_{11} + b_{22}}(|z_1|^2 + |z_2|^2) \le b_{11}|z_1|^2 + 2b_{12}\,\mathrm{Re}\{z_1\overline{z_2}\} + b_{22}|z_2|^2 \le C(|z_1|^2 + |z_2|^2), \tag{2.2}$$

for all $z_1, z_2 \in \mathbb{C}$.

Proof. We start the proof by demonstrating the validity of the first inequality. Using Young inequality

$$b_{11}|z_1|^2 + 2b_{12}\mathrm{Re}\{z_1\overline{z_2}\} + b_{22}|z_2|^2 \ge b_{11}|z_1|^2 - \frac{b_{12}^2}{b_{22}}|z_1|^2, \tag{2.3}$$

and

$$b_{11}|z_1|^2 + 2b_{12}\mathrm{Re}\{z_1\overline{z_2}\} + b_{22}|z_2|^2 \ge b_{22}|z_1|^2 - \frac{b_{12}^2}{b_{11}}|z_2|^2. \tag{2.4}$$

Multiplying (2.3) by b_{22} and (2.4) by b_{11}, we get

$$b_{11}b_{22}|z_1|^2 + 2b_{22}b_{12}\mathrm{Re}\{z_1\overline{z_2}\} + b_{22}^2|z_2|^2 \ge b_{11}b_{22}|z_1|^2 - b_{12}^2|z_1|^2, \tag{2.5}$$

and

$$b_{11}^2|z_1|^2 + 2b_{11}b_{12}\mathrm{Re}\{z_1\overline{z_2}\} + b_{11}b_{22}|z_2|^2 \geq b_{11}b_{22}|z_2|^2 - b_{12}^2|z_2|^2. \tag{2.6}$$

Adding (2.5) and (2.6)

$$b_{11}|z_1|^2 + 2b_{12}\mathrm{Re}\{z_1\overline{z_2}\} + b_{22}|z_2|^2 \geq \frac{(b_{11}b_{22} - b_{12}^2)}{b_{11} + b_{22}}(|z_1|^2 + |z_2|^2).$$

This concludes the proof of the first inequality in (2.2).

On the other hand, using Cauchy-Schwarz inequality

$$b_{11}|z_1|^2 + 2b_{12}\,\mathrm{Re}\{z_1\overline{z_2}\} + b_{22}|z_2|^2 \leq b_{11}|z_1|^2 + 2b_{12}\,|z_1||z_2| + b_{22}|z_2|^2. \tag{2.7}$$

Using Young inequality

$$|z_1||z_2| \leq \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2. \tag{2.8}$$

Then, replacing (2.8) into (2.7), we have

$$\begin{aligned} b_{11}|z_1|^2 + 2b_{12}\,\mathrm{Re}\{z_1\overline{z_2}\} + b_{22}|z_2|^2 &\leq b_{11}|z_1|^2 + b_{12}(|z_1|^2 + |z_2|^2) + b_{22}|z_2|^2 \\ &\leq (b_{11} + b_{12})|z_1|^2 + (b_{22} + b_{12})|z_2|^2 \\ &\leq C(|z_1|^2 + |z_2|^2), \end{aligned}$$

where $C = \max\left\{b_{11} + b_{12},\ b_{22} + b_{12}\right\}$, which proves the second inequality in (2.2).

The proof is complete. □

Note 13. If B is a positive definite matrix, then, by Theorem 2.50, the constant $\frac{\det(B)}{b_{11}+b_{22}}$ in (2.2) is positive.

3 WELL - POSEDENESS

3.1 THE PROBLEM

The aim of this chapter is to investigate the existence and uniqueness of solutions for the following Timoshenko system using the Lumer-Philips Theorem (Theorem 2.45). Thus, let us examine the set of equations

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + b_{11}\varphi_t + b_{12}\psi_t = 0 \text{ in } (0,l)\times(0,\infty), \tag{3.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + b_{21}\varphi_t + b_{22}\psi_t = 0 \text{ in } (0,l)\times(0,\infty), \tag{3.2}$$

with initial conditions

$$\varphi(\cdot,0) = \varphi_0(\cdot),\ \varphi_t(\cdot,0) = \varphi_1(\cdot), \psi(\cdot,0) = \psi_0(\cdot), \psi_t(\cdot,0) = \psi_1(\cdot), \tag{3.3}$$

and Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(l,t) = \psi(0,t) = \psi(l,t) = 0, \ \ t \geq 0, \tag{3.4}$$

where $\rho_1, \rho_2, b, k > 0$.

3.2 SEMIGROUP FORMULATION

In order to use the linear semigroup theory, we shall consider the notations

$$\Phi = \varphi_t, \ \ \Psi = \psi_t \ \text{ and } \ \mathbf{U} = (\varphi, \Phi, \psi, \Psi)^\top,$$

where, superscript $\top$ denotes the vector transpose operation.

Now, from the equations (3.1) and (3.2), we get

$$\mathbf{U}_t = \begin{bmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{b_{11}}{\rho_1}\Phi - \frac{b_{12}}{\rho_1}\Psi \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) - \frac{b_{21}}{\rho_2}\Phi - \frac{b_{22}}{\rho_2}\Psi \end{bmatrix} := \mathcal{A}\mathbf{U} \tag{3.5}$$

and, by (3.3)

$$\mathbf{U}(t=0) = (\varphi_0, \varphi_1, \psi_0, \psi_1)^\top := \mathbf{U}_0.$$

So, it is possible to write the intial-boundary value problem (3.1)-(3.4) as the following abstract Cauchy problem

$$\begin{aligned} &\mathbf{U}_t = \mathcal{A}\mathbf{U},\ t > 0, \\ &\mathbf{U}(0) = \mathbf{U}_0. \end{aligned} \tag{3.6}$$

To approach (3.6) including the boundary conditions (3.4), we define space

$$\mathcal{H} = H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l). \tag{3.7}$$

The space $\mathcal{H}$ is a Hilbert space with inner product $(\cdot,\cdot) : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C}$ defined by

$$\begin{aligned} (\mathbf{U},\widetilde{\mathbf{U}}) &= \int_0^l \Phi\overline{\widetilde{\Phi}}\, dx + \int_0^l \Psi\overline{\widetilde{\Psi}}\, dx + \int_0^l \psi_x\overline{\widetilde{\psi_x}}\, dx + \int_0^l \varphi_x\widetilde{\varphi_x}\, dx \\ &\equiv (\Phi,\widetilde{\Phi})_2 + (\Psi,\widetilde{\Psi})_2 + (\psi_x,\widetilde{\psi_x})_2 + (\varphi_x,\widetilde{\varphi_x})_2, \end{aligned} \tag{3.8}$$

where $\mathbf{U} = (\varphi,\Phi,\psi,\Psi)^\top$, $\widetilde{\mathbf{U}} = (\widetilde{\varphi},\widetilde{\Phi},\widetilde{\psi},\widetilde{\Psi})^\top$ and $(\cdot,\cdot)_2 = (\cdot,\cdot)_{L^2(0,l)}$. The inner product (3.8) induces the following norm $\|\cdot\| : \mathcal{H} \longrightarrow \mathbb{R}^+$

$$\begin{aligned} \|\mathbf{U}\|^2 = (\mathbf{U},\mathbf{U}) &= \int_0^l \Phi\overline{\Phi}\, dx + \int_0^l \Psi\overline{\Psi}\, dx + \int_0^l \psi_x\overline{\psi_x}\, dx + \int_0^l \varphi_x\overline{\varphi_x}\, dx \\ &\equiv \|\Phi\|_2^2 + \|\Psi\|_2^2 + \|\psi_x\|_2^2 + \|\varphi_x\|_2^2, \end{aligned}$$

where $\|\cdot\|_2 = \|\cdot\|_{L^2(0,l)}$.

In space $\mathcal{H}$, we also consider the function $(\cdot,\cdot)_{\mathcal{H}} : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C}$ defined by

$$(\mathbf{U},\widetilde{\mathbf{U}})_{\mathcal{H}} = \rho_1(\Phi,\widetilde{\Phi})_2 + \rho_2(\Psi,\widetilde{\Psi})_2 + b(\psi_x,\widetilde{\psi_x})_2 + k(\varphi_x+\psi,\widetilde{\varphi}_x+\widetilde{\psi})_2. \tag{3.9}$$

This function induces the following map $\|\cdot\|_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathbb{R}^+$ defined as

$$\|\mathbf{U}\|_{\mathcal{H}}^2 = (\mathbf{U},\mathbf{U})_{\mathcal{H}} = \rho_1\|\Phi\|_2^2 + \rho_2\|\Psi\|_2^2 + b\|\psi_x\|_2^2 + k\|\varphi_x+\psi\|_2^2. \tag{3.10}$$

Lemma 3.1. *The map $(\cdot,\cdot)_{\mathcal{H}} : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C}$ defined in* (3.9) *is an inner product in $\mathcal{H}$.*

Proof. Let $\mathbf{U} \in \mathcal{H}$, such that, $(\mathbf{U},\mathbf{U})_{\mathcal{H}} = 0$. The definition (3.9) implies that

$$\Phi = \Psi = \psi_x = \varphi_x + \psi = 0.$$

Now, Poincaré Inequality implies that $\psi = \varphi = 0$. Therefore, $\mathbf{U} = 0$. The remaining properties in Definition 2.9 can be inferred from the properties satisfied by the inner product in $L^2(0,l)$. □

The following result is a consequence of Lemma 3.1.

Lemma 3.2. *The map $\|\cdot\|_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathbb{R}^+$ defined in* (3.10) *is a norm in $\mathcal{H}$.*

Lemma 3.3. *The norms* $\|\cdot\|$ *and* $\|\cdot\|_{\mathcal{H}}$ *are equivalent.*

Proof. Firstly, observe that

$$\begin{aligned}\|\mathbf{U}\|_{\mathcal{H}}^2 &= \rho_1\|\Phi\|_2^2 + \rho_2\|\Psi\|_2^2 + b\|\psi_x\|_2^2 + k\|\varphi_x + \psi\|_2^2 \\ &\leq C_1\left(\|\Phi\|_2^2 + \|\Psi\|_2^2 + \|\psi_x\|_2^2 + \|\varphi_x\|_2^2\right) \\ &= C_1\|\mathbf{U}\|^2.\end{aligned}$$

where $C_1 = \max\left\{\rho_1, \rho_2, b, k\right\}$Secondly,

$$\begin{aligned}\|\mathbf{U}\|^2 &= \|\Phi\|_2^2 + \|\Psi\|_2^2 + \|\psi_x\|_2^2 + \|\varphi_x + \psi - \psi\|_2^2 \\ &\leq C_2\left(\rho_1\|\Phi\|_2^2 + \rho_2\|\Psi\|_2^2 + b\|\psi_x\|_2^2 + k\|\varphi_x + \psi\|_2^2\right) \\ &= C_2\|\mathbf{U}\|_{\mathcal{H}}^2.\end{aligned}$$

where $C_2 = \max\left\{\frac{1}{\rho_1}, \frac{1}{\rho_2}, \frac{1+c_p}{b}, \frac{1}{k}\right\}$. The above inequalities imply that $\|\cdot\|$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent norms in $\mathcal{H}$.

□

Once we defined the phase space $\mathcal{H}$, we can now define the domain of the operator $\mathcal{A}$ via the following proposition.

Proposition 3.1. *The domain of the operator* $\mathcal{A}$ *defined in* (3.5) *is given by*

$$D(\mathcal{A}) = \{\mathbf{U} \in \mathcal{H} \mid \varphi, \psi \in H^2(0,l) \cap H_0^1(0,l), \Phi, \Psi \in H_0^1(0,l)\}.$$

Proof. Firstly, remember that, from the Definition 2.43, the domain of operator $\mathcal{A}$ is given by $\Lambda = \{\mathbf{U} \in \mathcal{H} \mid \mathcal{A}\mathbf{U} \in \mathcal{H}\}$. The inclusion $D(\mathcal{A}) \subset \Lambda$ is satisfied. Now, let $\mathbf{U} \in \Lambda$ and therefore

$$\begin{aligned}\Phi, \Psi &\in H_0^1(0,l), \\ \frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{b_{11}}{\rho_1}\Phi - \frac{b_{12}}{\rho_1}\Psi &\in L^2(0,l), \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) - \frac{b_{21}}{\rho_2}\Phi - \frac{b_{22}}{\rho_2}\Psi &\in L^2(0,l).\end{aligned}$$

Note that,

$$\frac{k}{\rho_1}\varphi_{xx} = \left(\frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{b_{11}}{\rho_1}\Phi - \frac{b_{12}}{\rho_1}\Psi\right) - \frac{k}{\rho_1}(\psi_x) + \frac{b_{11}}{\rho_1}\Phi + \frac{b_{12}}{\rho_1}\Psi,$$

and this implies that $\varphi_{xx} \in L^2(0,l)$. Also,

$$\frac{b}{\rho_2}\psi_{xx} = \left(\frac{k}{\rho_2}(\varphi_x + \psi) - \frac{b_{21}}{\rho_2}\Phi - \frac{b_{22}}{\rho_2}\Psi\right) + \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{b_{21}}{\rho_2}\Phi + \frac{b_{22}}{\rho_2}\Psi,$$

and, thus $\psi_{xx} \in L^2(0,l)$.

Consequently, $\varphi, \psi \in H^2(0,l) \cap H_0^1(0,l)$. Therefore, $\Lambda \subset D(\mathcal{A})$.

□

3.3 Existence and Uniqueness

This section is focused in showing, the existence and uniqueness of solution for the problem (3.1)-(3.4). To demonstrate the existence and uniqueness result, we will consider the following condition for the matrix B, which we call damping matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}. \tag{3.11}$$

Condition 01: A real matrix B, as defined in (3.11), satisfies Condition 01 if

$$b_{11}|z_1|^2 + (b_{21} + b_{12})\text{Re}\,\{z_1\overline{z_2}\} + b_{22}|z_2|^2 \geq 0, \ \forall\ z_1, z_2 \in \mathbb{C}. \tag{3.12}$$

Next, we have some examples of matrices that satisfy the Condition 01.

Example 1. Positively defined matrices satisfy Condition 01. See Definition 2.49, Theorem 2.50 and Lemma 2.51 (Note 13).

Example 2. The identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that, $b_{11}|z_1|^2 + (b_{21} + b_{12})\text{Re}\,\{z_1\overline{z_2}\} + b_{22}|z_2|^2 = |z_1|^2 + |z_2|^2 \geq 0$, for all $z_1, z_2 \in \mathbb{C}$.

Example 3. The matrix B given by

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}.$$

Note that, $b_{11}|z_1|^2 + (b_{21} + b_{12})\text{Re}\,\{z_1\overline{z_2}\} + b_{22}|z_2|^2 = 2|z_1|^2 + 3|z_2|^2 \geq 0$. Therefore, Condition 01 is satisfied.

Example 4. The matrix B given by

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that, $b_{11}|z_1|^2 + (b_{21} + b_{12})\text{Re}\,\{z_1\overline{z_2}\} + b_{22}|z_2|^2 = |z_2|^2 \geq 0$. Therefore, Condition 01 is satisfied.

The main result of this chapter is the following theorem.

Theorem 3.4. *Let* $\rho_1, \rho_2, b, k > 0$. *If* $\mathbf{U}_0 \in D(\mathcal{A})$, *then the Cauchy problem* (3.6) *admits a unique solution* $\mathbf{U} \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H})$, *under Condition 01.*

Proof. By the Theorem 2.44, it is enough to show that the operator $\mathcal{A}$ (defined in (3.5)) is the infinitesimal generator of a contraction C_0-semigroup in $\mathcal{H}$. By Lumer-Philips Theorem (Theorem 2.45), is sufficient showing that

(i) $\overline{D(\mathcal{A})} = \mathcal{H}$;

(ii) $\mathcal{A}$ is dissipative in $\mathcal{H}$, i.e., $\text{Re}(\mathcal{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}} \leq 0$, for all $\mathbf{U} \subset D(A)$;

(iii) $I_{\mathcal{H}} - \mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is surjective.

We will initially provide the proofs for items (ii) and (iii), leaving the proof of (i) for the final part.

Proof of (ii).

Let $\mathbf{U} \in D(\mathcal{A})$ and

$$\mathcal{A}\mathbf{U} = \begin{bmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{b_{11}}{\rho_1}\Phi - \frac{b_{12}}{\rho_1}\Psi \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) - \frac{b_{21}}{\rho_2}\Phi - \frac{b_{22}}{\rho_2}\Psi \end{bmatrix}.$$

By the inner product defined in (3.9), we obtain

$$\begin{aligned}
(\mathcal{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}} &= \rho_1 \int_0^l \Big(\frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{b_{11}}{\rho_1}\Phi - \frac{b_{12}}{\rho_1}\Psi\Big)\overline{\Phi}\,dx + \rho_2 \int_0^l \Big(\frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) \\
&\quad - \frac{b_{21}}{\rho_2}\Phi - \frac{b_{22}}{\rho_2}\Psi\Big)\overline{\Psi}\,dx + b\int_0^l \Psi_x\overline{\psi_x}\,dx + k\int_0^l (\Phi_x + \Psi)\overline{(\varphi_x + \psi)}\,dx \\
&= k\int_0^l (\varphi_x + \psi)_x\overline{\Phi}\,dx - b_{11}\int_0^l \Phi\overline{\Phi}\,dx - b_{12}\int_0^l \Psi\overline{\Phi}\,dx + b\int_0^l \psi_{xx}\overline{\Psi}\,dx \\
&\quad - k\int_0^l (\varphi_x + \psi)\overline{\Psi}\,dx - b_{21}\int_0^l \Phi\overline{\Psi}\,dx - b_{22}\int_0^l \Psi\overline{\Psi}\,dx + b\int_0^l \Psi_x\overline{\psi_x}\,dx \\
&\quad + k\int_0^l \left(\Phi_x\overline{\varphi_x} + \Phi_x\overline{\psi} + \Psi\overline{\varphi_x} + \Psi\overline{\psi}\right)\,dx. \qquad (3.13)
\end{aligned}$$

Integrating by parts the following terms, we find

$$k\int_0^l (\varphi_x + \psi)_x\overline{\Phi}\,dx = -k\int_0^l (\varphi_x + \psi)\overline{\Phi_x}\,dx = -k(\varphi_x, \Phi_x)_2 - k(\psi, \Phi_x)_2,$$

and

$$-b\int_0^l \psi_{xx}\overline{\Psi}\,dx = b\int_0^l \psi_x\overline{\Psi_x}\,dx = -b(\psi_x, \Psi_x)_2.$$

Then, replacing in (3.13)

$$
\begin{aligned}
(\mathcal{A}\mathbf{U},\mathbf{U})_{\mathcal{H}} = & -k(\varphi_x,\Phi_x)_2 - k(\psi,\Phi_x)_2 - b_{11}\|\Phi\|_2^2 - b_{12}(\Psi,\Phi)_2 - b(\psi_x,\Psi_x)_2 \\
& - k(\varphi_x,\Psi)_2 - k(\psi,\Psi)_2 \,-\, b_{21}(\Phi,\Psi)_2 - b_{22}\|\Psi\|_2^2 \,+\, b(\Psi_x,\psi_x)_2 \\
& + k(\Phi_x,\varphi_x)_2 + k(\Phi_x,\psi)_2 + k(\Psi,\varphi_x)_2 + k(\Psi,\psi)_2.
\end{aligned}
$$

The Condition 01 (3.12) implies

$$
\mathrm{Re}(\mathcal{A}\mathbf{U},\mathbf{U})_{\mathcal{H}} = -b_{11}\|\Phi\|_2^2 - \mathrm{Re}(b_{21}+b_{12})(\Psi,\Phi)_2 - b_{22}\|\Psi\|_2^2 \le 0. \tag{3.14}
$$

Hence, $\mathcal{A}$ is dissipative operator.

Proof of (iii). Given $\mathbf{F} = (f_1, f_2, f_3, f_4)^\top \in \mathcal{H}$, we will prove that the resolvent equation $(I_{\mathcal{H}} - \mathcal{A})\mathbf{U} = \mathbf{F}$ has a unique solution $\mathbf{U} \in D(\mathcal{A})$. Rewriting on terms of its coordinates, we obtain the following system

$$
\varphi - \Phi = f_1 \ \text{ in } \ H_0^1(0,l), \tag{3.15}
$$

$$
\Phi - \frac{k}{\rho_1}(\varphi_x+\psi)_x + \frac{b_{11}}{\rho_1}\Phi + \frac{b_{12}}{\rho_1}\Psi = f_2 \ \text{ in } \ L^2(0,l), \tag{3.16}
$$

$$
\psi - \Psi = f_3 \ \text{ in } \ H_0^1(0,l), \tag{3.17}
$$

$$
\Psi - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x+\psi) + \frac{b_{21}}{\rho_2}\Phi + \frac{b_{22}}{\rho_2}\Psi = f_4 \ \text{ in } \ L^2(0,l). \tag{3.18}
$$

By (3.15), $\Phi = \varphi - f_1$ and by (3.17), $\Psi = \psi - f_3$. Replacing those at (3.16) and (3.18) we have

$$
\varphi - f_1 - \frac{k}{\rho_1}(\varphi_x+\psi)_x + \frac{b_{11}}{\rho_1}(\varphi - f_1) + \frac{b_{12}}{\rho_1}(\psi - f_3) = f_2
$$

then

$$
(\rho_1 + b_{11})\varphi - k(\varphi_x+\psi)_x + b_{12}\psi = \rho_1(f_1+f_2) + b_{11}f_1 + b_{12}f_3.
$$

Also

$$
\psi - f_3 - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x+\psi) + \frac{b_{21}}{\rho_2}(\varphi - f_1) + \frac{b_{22}}{\rho_2}(\psi - f_3) = f_4
$$

and so,

$$
(\rho_2 - b_{22})\psi - b\psi_{xx} - k(\varphi_x+\psi) - b_{21}\varphi = \rho_2(f_3+f_4) + b_{21}f_1 + b_{22}f_3.
$$

From where we obtain

$$(\rho_1 + b_{11})\varphi - k(\varphi_x + \psi)_x + b_{12}\psi = \rho_1(f_1 + f_2) + b_{11}f_1 + b_{12}f_3, \tag{3.19}$$
$$(\rho_2 - b_{22})\psi - b\psi_{xx} - k(\varphi_x + \psi) - b_{21}\varphi = \rho_2(f_3 + f_4) + b_{21}f_1 + b_{22}f_3. \tag{3.20}$$

Now, defining

$$g_1 \equiv \rho_1(f_1 + f_2) + b_{11}f_1 + b_{12}f_3, \tag{3.21}$$
$$g_2 \equiv \rho_2(f_3 + f_4) + b_{21}f_1 + b_{22}f_3, \tag{3.22}$$

then, the system (3.19)-(3.20) can be written as

$$(\rho_1 + b_{11})\varphi - k(\varphi_x + \psi)_x + b_{12}\psi = g_1, \tag{3.23}$$
$$(\rho_2 - b_{22})\psi - b\psi_{xx} - k(\varphi_x + \psi) - b_{21}\varphi = g_2, \tag{3.24}$$

which we will solve through two stages.

1^{st}**stage.**

Affirmation: There is a single pair $(\varphi, \psi) \in H_0^1(0,l) \times H_0^1(0,l)$ which satisfies the variational equation

$$\int_0^l [(\rho_1 + b_{11})\varphi\overline{\widetilde{\varphi}} + (\rho_2 + b_{22})\psi\overline{\widetilde{\psi}} + b\psi_{xx}\overline{\widetilde{\psi}} + k(\varphi_x + \psi)\overline{(\widetilde{\varphi}_x + \widetilde{\psi})} + b_{12}\psi\overline{\widetilde{\psi}} + b_{21}\varphi\overline{\widetilde{\varphi}}]\, dx$$
$$= \int_0^l [g_1\overline{\widetilde{\varphi}} + g_2\overline{\widetilde{\psi}}]\ dx. \tag{3.25}$$

Firstly, we define the following function

$$a : (H_0^1(0,l) \times H_0^1(0,l))^2 \longrightarrow \mathbb{C}$$
$$((\varphi, \psi), (\widetilde{\varphi}, \widetilde{\psi})) \longmapsto a((\varphi, \psi), (\widetilde{\varphi}, \widetilde{\psi})),$$

where

$$a((\varphi, \psi), (\widetilde{\varphi}, \widetilde{\psi})) = \int_0^l [(\rho_1 + b_{11})\varphi\overline{\widetilde{\varphi}} + (\rho_2 + b_{22})\psi\overline{\widetilde{\psi}} + b\psi_{xx}\overline{\widetilde{\psi}} + k(\varphi_x + \psi)\overline{(\widetilde{\varphi}_x + \widetilde{\psi})}\, dx,$$

and

$$h : H_0^1(0,l) \times H_0^1(0,l) \longrightarrow \mathbb{C}$$
$$(\widetilde{\varphi}, \widetilde{\psi}) \longmapsto h(\widetilde{\varphi}, \widetilde{\psi}),$$

where

$$h(\widetilde{\varphi},\widetilde{\psi}) = \int_0^l \left(g_1\overline{\widetilde{\varphi}} + g_2\overline{\widetilde{\psi}} \right) dx.$$

Thus defined, a is a sesquilinear form, continuous and coercive.

- **a is continuous.**

  Indeed, by using the triangle, Cauchy-Schwarz and Poincaré inequalities, we find

$$\begin{aligned}
&|a((\varphi,\psi),(\widetilde{\varphi},\widetilde{\psi}))| \\
&= \left| \int_0^l \Big((\rho_1+b_{11})\varphi\overline{\widetilde{\varphi}} + (\rho_2+b_{22})\psi\overline{\widetilde{\psi}} + b\psi_x\overline{\widetilde{\psi_x}} + k(\varphi_x+\psi)\overline{(\widetilde{\varphi_x}+\widetilde{\psi})} \Big) \, dx \right| \\
&\le |\rho_1+b_{11}| \, \|\varphi\|_2\|\widetilde{\varphi}\|_2 + |\rho_2+b_{22}| \, \|\psi\|_2\|\widetilde{\psi}\|_2 + b\,\|\psi_x\|_2\|\widetilde{\psi_x}\|_2 \\
&\quad + k\|\varphi_x+\psi\|_2\|\widetilde{\varphi_x}+\widetilde{\psi}\|_2 \\
&\le |\rho_1+b_{11}|c_p\|\varphi_x\|_2\|\widetilde{\varphi_x}\|_2 + |\rho_2+b_{22}|c_p\|\psi_x\|_2\|\widetilde{\psi_x}\|_2 + b\|\psi_x\|_2\|\widetilde{\psi_x}\|_2 \\
&\quad + k(\,\|\varphi_x\|_2 + c_p\|\psi_x\|_2)(\|\widetilde{\varphi_x}\|_2 + c_p\|\widetilde{\psi}_x\|_2) \\
&= C_3\Big(\|\varphi_x\|_2\|\widetilde{\varphi_x}\|_2 + \|\psi_x\|_2\|\widetilde{\psi_x}\|_2 + \|\varphi_x\|_2\|\widetilde{\psi_x}\|_2 + \|\widetilde{\varphi_x}\|_2\|\psi_x\|_2 \Big) \\
&\le C_3\|(\varphi,\psi)\|_{H_0^1\times H_0^1}\|(\widetilde{\varphi},\widetilde{\psi})\|_{H_0^1\times H_0^1},
\end{aligned}$$

where $C_3 = \max\Big\{|\rho_1+b_{11}|c_p, |p_2+b_{22}|c_p, k, b, kc_p, kc_p^2\Big\}$. As established before in Theorem 2.35, we denote c_p as the Poincaré constant. Hence, a is continuous.

- **a is coercive.**

  Note that

$$\begin{aligned}
\|(\varphi,\psi)\|^2_{H_0^1\times H_0^1} &= \|\varphi_x\|_2^2 + \|\psi_x\|_2^2 \\
&= \|\varphi_x+\psi-\psi\|_2^2 + \|\psi_x\|_2^2 \\
&\le \|\varphi_x+\psi\|_2^2 + \|\psi\|_2^2 + \|\psi_x\|_2^2 \\
&\le (\rho_1+b_{11})\|\varphi\|_2^2 + \frac{1}{k}k\|\varphi_x+\psi\|_2^2 + \frac{1}{(\rho_2+b_{22})}(\rho_2+b_{22})\|\psi\|_2^2 \\
&+ \frac{1}{b}b\|\psi_x\|_2^2 \\
&\le C_4\, a((\varphi,\psi),(\varphi,\psi)),
\end{aligned}$$

where $C_4 = \max\Big\{1, \frac{1}{k}, \frac{1}{(\rho_2+b_{22})}, \frac{1}{b}\Big\}$. Then,

$$a((\varphi,\psi),(\varphi,\psi)) \ge \frac{1}{C_4}\|(\varphi,\psi)\|^2_{H_0^1\times H_0^1}.$$

Therefore, a is coercive.

- h **is bounded.**

Using the Triangle, Poincaré and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned} |h(\widetilde{\varphi},\widetilde{\psi})| &\leq \left|\int_0^l g_1\overline{\widetilde{\varphi}}\,dx\right| + \left|\int_0^l g_2\overline{\widetilde{\psi}}\,dx\right| \\ &\leq c_p\|g_1\|_2\|\widetilde{\varphi_x}\|_2 + c_p\|g_2\|_2\|\widetilde{\psi_x}\|_2 \\ &\leq C_5\|(\widetilde{\varphi},\widetilde{\psi})\|_{H_0^1\times H_0^1}, \end{aligned}$$

where $C_5 = \max\{c_p\|g_1\|_2, c_p\|g_2\|_2\}$. Hence, h is bounded.

Therefore, from the Lax-Milgram Theorem (Theorem 2.14), there is a single pair $(\varphi,\psi) \in H_0^1(0,l) \times H_0^1(0,l)$ such that

$$a((\varphi,\psi),(\widetilde{\varphi},\widetilde{\psi})) = h(\widetilde{\varphi},\widetilde{\psi}).$$

2^{nd} **stage.**

Show that $(\varphi,\psi) \in H^2(0,l) \times H^2(0,l)$ and satisfies (3.23)-(3.24). Indeed. Consider $\widetilde{\varphi} \in C_0^1(0,l)$ and $\widetilde{\psi} = 0$, applying in (3.25)

$$\int_0^l \varphi_x\overline{\widetilde{\varphi_x}}\,dx = -\frac{1}{k}\left(\int_0^l \Big((\rho_1+b_{11})\varphi + k\psi_x + b_{21}\varphi - g_1\Big)\overline{\widetilde{\varphi}}\,dx\right). \tag{3.26}$$

As $\varphi_x, (\rho_1+b_{21})\varphi + b_{21}\varphi - g_1 \in L^2(0,l)$ and (3.26) holds, then by the definition of weak derivatives, $\varphi_x \in H_0^1(0,l)$, that is, $\varphi \in H^2(0,l)$. And still,

$$k\varphi_{xx} = (\rho_1+b_{11})\varphi + k\psi_x + b_{21}\varphi - g_1 \ \text{ in } \ L^2(0,l).$$

Also, remembering g_1 in (3.21) and taking $\Phi = \varphi - f_1 \in H_0^1(0,l)$, you get

$$\Phi - \frac{k}{\rho_1}(\varphi_x+\psi)_x + \frac{b_{11}}{\rho_1}\Phi + \frac{b_{12}}{\rho_1}\Psi = f_2.$$

Therefore, (3.16) is satisfied.

On the other hand, replacing in (3.25) $\widetilde{\psi} \in C_0^1(0,l)$ and $\widetilde{\varphi} = 0$, we have

$$\int_0^l \psi_x\overline{\widetilde{\psi_x}}\,dx = -\frac{1}{b}\left(\int_0^l \Big((\rho_2+b_{22})\psi + k(\varphi_x+\psi) + b_{12}\psi - g_2\Big)\overline{\widetilde{\psi}}\,dx\right).$$

Since $\psi_x, (\rho_2+b_{22})\psi + k(\varphi_x+\psi) + b_{12}\psi - g_2 \in L^2(0,l)$ and (3.22) holds, then by the definition of weak derivative, $\psi_x \in H^1(0,l)$, from where $\psi \in H^2(0,l)$. And also,

$$b\psi_{xx} = (\rho_2+b_{22})\psi + k(\varphi_x+\psi) + b_{12}\psi - g_2 \ \text{ in } \ L^2(0,l).$$

Furthermore, from (3.22) and $\Psi = \psi - f_3 \in H_0^1(0,l)$, we obtain

$$\Psi - \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{b_{21}}{\rho_2}\Phi + \frac{b_{22}}{\rho_2}\Psi = f_4 \text{ in } L^2(0,l).$$

Thus, (3.18) is satisfied.

Proof of (i): We have shown that $\mathcal{A}$ is dissipative operator and $I_{\mathcal{H}} - \mathcal{A}$ is surjective. Then, $\mathcal{H}$ being a Hilbert space and by the Theorems 2.16 and 2.18 it comes that $\overline{D(\mathcal{A})} = \mathcal{H}$. □

4 EXPONENTIAL STABILITY - B POSITIVE DEFINITE

Throughout this chapter, we will show that the Cauchy problem (3.1)-(3.4) is exponentially stable via Theorem 2.47.

Starting now, the symbol C will denote various positive constants dependent on the parameters $\rho_1, \rho_2, b, k, c_p$, and b_{ij}, where $i, j = 1, 2$. If they depend on a specific parameter η, we denote them as C_η.

Lemma 4.1. *Let* $\rho_1, \rho_2, b, k > 0$ *and* $b_{ij} \in \mathbb{R}$. *Then,* $0 \in \rho(\mathcal{A})$.

Proof. Showing that $0 \in \rho(\mathcal{A})$, by Definition 2.20, is the same as showing that $(-\mathcal{A})^{-1}$ exists and is a bounded operator. Given $\mathbf{F} \in \mathcal{H}$, we will show the resolvent equation

$$-\mathcal{A}\mathbf{U} = \mathbf{F}, \tag{4.1}$$

has a unique solution $\mathbf{U} \in D(\mathcal{A})$. Rewriting in terms of its components, we have

$$\begin{aligned} -\Phi &= f_1, && (4.2)\\ -k(\varphi_x + \psi_x) + b_{11}\Phi + b_{12}\Psi &= \rho_1 f_2, && (4.3)\\ -\Psi &= f_3, && (4.4)\\ b\psi_{xx} + k(\varphi_x + \psi) + b_{21}\Phi + b_{22}\Psi &= \rho_2 f_4. && (4.5) \end{aligned}$$

It follows from (4.2) and (4.4) that

$$\Phi = -f_1 \ \text{ and } \ \Psi = -f_3.$$

Replacing into (4.3) and (4.4), respectively, we have

$$-k(\varphi_x + \psi)_x = \rho_1 f_2 + b_{11} f_1 + b_{12} f_3,$$

and

$$b\psi_{xx} + k(\varphi_x + \psi) = \rho_2 f_4 + b_{21} f_1 + b_{22} f_3.$$

Now, defining

$$\begin{aligned} g_3 &\equiv \rho_1 f_2 + b_{11} f_1 + b_{12} f_3,\\ g_4 &\equiv \rho_2 f_4 + b_{21} f_1 + b_{22} f_3. \end{aligned}$$

we obtain the following system

$$\begin{aligned} -k(\varphi_x+\psi)_x &= g_1 \ \text{ in } \ L^2(0,l), \\ b\psi_{xx}+k(\varphi_x+\psi) &= g_2 \ \text{ in } \ L^2(0,l). \end{aligned}$$

Note that, as proven before in Theorem 3.4, the system above has a unique solution. From now on, we will focus on showing $(-\mathcal{A})^{-1}$ is bounded, as $(-\mathcal{A})^{-1}\mathbf{F} = \mathbf{U}$, it is sufficient to show that there exists a positive constant C, such that

$$\|(-\mathcal{A})^{-1}\mathbf{F}\|_{\mathcal{H}} = \|\mathbf{U}\|_{\mathcal{H}} \le C\|\mathbf{F}\|_{\mathcal{H}}.$$

Note that, using (4.2) and by Poincaré inequality, we have

$$\begin{aligned} \|\Phi\|_2^2 = \|f_1\|_2^2 \le c_p^2\|f_{1,x}\|_2^2 &\le c_p^2(\|f_{1,x}+f_3\|_2+\|f_3\|_2)^2 \\ &\le 2c_p^2\|f_{1,x}+f_3\|_2^2+2c_p^4\|f_{3,x}\|_2^2 \\ &\le C\|\mathbf{F}\|_{\mathcal{H}}. \end{aligned} \tag{4.6}$$

where $C=\max\left\{2c_p^2, 2c_p^4\right\}$. Similarly, from (4.4), we have

$$\|\Psi\|_2^2 = \|f_3\|_2^2 \le c_p^2\|f_{3,x}\|_2^2 \le C\|\mathbf{F}\|_{\mathcal{H}}. \tag{4.7}$$

Taking the inner product of (4.3) with $\overline{\varphi}$ in $L^2(0,l)$, we obtain

$$-k\int_0^l(\varphi_x+\psi)_x\overline{\varphi}\,dx + b_{11}\int_0^l\Phi\overline{\varphi}\,dx + b_{12}\int_0^l\Psi\overline{\varphi}\,dx \quad = \quad \rho_1\int_0^l f_2,\overline{\varphi}\,dx.$$

Integrating by parts,

$$k\int_0^l(\varphi_x+\psi)\overline{\varphi_x}\,dx + b_{11}\int_0^l\Phi\overline{\varphi}\,dx + b_{12}\int_0^l\Psi\overline{\varphi}\,dx = \rho_1\int_0^l f_2\overline{\varphi}\,dx.$$

Also, taking the inner product in $L^2(0,l)$ of the resolvent equation (4.5) with $\overline{\psi}$, we find

$$-b\int_0^l\psi_{xx}\overline{\psi}\,dx + k\int_0^l(\varphi_x+\psi)\overline{\psi}\,dx + b_{21}\int_0^l\Phi\overline{\psi}\,dx + b_{22}\int_0^l\Psi\overline{\psi}\,dx = \rho_2\int_0^l f_4\overline{\psi}\,dx. \tag{4.8}$$

Integrating by parts,

$$b\int_0^l\psi_x\overline{\psi_x}\,dx + k\int_0^l(\varphi_x+\psi)\overline{\psi}\,dx + b_{21}\int_0^l\Phi\overline{\psi}\,dx + b_{22}\int_0^l\Psi\overline{\psi}\,dx = \rho_2\int_0^l f_4\overline{\psi}\,dx. \tag{4.9}$$

Then, adding (4.8) and (4.9), we have

$$\begin{aligned} k\|\varphi_x+\psi\|_2^2+b\|\psi_x\|_2^2 = & -b_{11}(\Phi,\varphi)_2 - b_{12}(\Psi,\varphi)_2 - b_{21}(\Phi,\psi)_2 \\ & - b_{22}(\Psi,\psi)_2 + \rho_1(f_2,\varphi)_2 + \rho_2(f_4,\psi)_2. \end{aligned}$$

Then, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} k\|\varphi_x+\psi\|_2^2+b\|\psi_x\|_2^2 \;\leq\; & |b_{11}|\|\Phi\|_2\|\varphi\|_2 + |b_{12}|\|\Psi\|_2\|\varphi\|_2 + |b_{21}|\|\Phi\|_2\|\psi\|_2 \\ & + |b_{22}|\|\Psi\|_2\|\psi\|_2 \;+\; \rho_2\|f_4\|_2\|\psi\|_2 + \rho_1\|f_2\|_2\|\varphi\|_2. \end{aligned}$$

Also, by Poincaré inequality, we get

$$\begin{aligned} k\|\varphi_x+\psi\|_2^2+b\|\psi_x\|_2^2 \;\leq\; & |b_{11}|c_p\|\Phi\|_2\|\varphi_x\|_2 + |b_{12}|c_p\|\Psi\|_2\|\varphi_x\|_2 + |b_{21}|c_p\|\Phi\|_2\|\psi_x\|_2 \\ & +|b_{22}|c_p\|\Psi\|_2\|\psi_x\|_2 \;+\; \rho_2 c_p\|f_4\|_2\|\psi_x\|_2 \;+\; \rho_1 c_p\|f_2\|_2\|\varphi_x\|_2 \\ \leq\; & |b_{11}|c_p\|\Phi\|_2\|\varphi_x+\psi\|_2 \;+\; |b_{11}|c_p^2\|\Phi\|_2\|\psi_x\|_2 \\ & +|b_{12}|c_p\|\Psi\|_2\|\varphi_x+\psi\|_2 + |b_{12}|c_p^2\|\Psi\|_2\|\psi_x\|_2 \\ & +|b_{21}|c_p\|\Phi\|_2\|\varphi_x+\psi\|_2 + |b_{21}|c_p\|\Phi\|_2^2\|\psi_x\|_2 \\ & +|b_{22}|\|\Psi\|_2\|\psi_x\|_2 + \rho_2 c_p\|f_4\|_2\|\psi_x\|_2 \\ & +\rho_1 c_p\|f_2\|_2\|\varphi_x+\psi\|_2 + \rho_1 c_p^2\|f_2\|_2\|\psi_x\|_2, \end{aligned}$$

where $C = \max\left\{|b_{11}|c_p, |b_{11}|c_p^2, |b_{12}|c_p, |b_{12}|c_p^2, |b_{21}|c_p, |b_{22}|, \rho_2 c_p, \rho_1 c_p, \rho_1 c_p^2\right\}$. Using Young inequality, with $\epsilon_1, \epsilon_2 > 0$, then exists constants $C_{\epsilon_1}, C_{\epsilon_2} > 0$, such that

$$\begin{aligned} k\|\varphi_x+\psi\|_2^2+b\|\psi_x\|_2^2 \;\leq\; & C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}} + \epsilon_1\|\varphi_x+\psi\|_2^2 + C_{\epsilon_1}\|\Phi\|_2^2 + C_{\epsilon_1}\|\Psi\|_2^2 \\ & +\epsilon_2\|\psi_x\|_2^2 + C_{\epsilon_2}\|\Phi\|_2^2 + C_{\epsilon_2}\|\Psi\|_2^2. \end{aligned}$$

Taking $\epsilon_1 = \frac{k}{2}$ and $\epsilon_2 = \frac{b}{2}$, we find from estimates (4.6), (4.7) that

$$b\|\psi_x\|_2^2 + k\|\varphi_x+\psi\|_2^2 \leq C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}, \tag{4.10}$$

for some $C > 0$. Therefore, we obtain from estimates (4.6), (4.7) and (4.10)

$$\|\mathbf{U}\|_{\mathcal{H}}^2 \leq C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}} + C\|\mathbf{F}\|_{\mathcal{H}}^2.$$

Applying, again, the Young inequality

$$\|\mathbf{U}\|_{\mathcal{H}}^2 \leq C\|\mathbf{F}\|_{\mathcal{H}}^2.$$

Hence, there is $C > 0$ such that

$$\|(-\mathcal{A})^{-1}\mathbf{F}\|_{\mathcal{H}} \leq C\|\mathbf{F}\|_{\mathcal{H}}, \ \forall \mathbf{F} \in \mathcal{H}.$$

Therefore, $0 \in \rho(\mathcal{A})$. □

Lemma 4.2. *Suppose that* $\rho_1, \rho_2, b, k > 0$ *and let* B *be a positive definite matrix. Then,* $i\mathbb{R} \subset \rho(\mathcal{A})$.

Proof. From the Theorem 2.41, we obtain that $H^2(0,l) \overset{c}{\hookrightarrow} H^1(0,l) \overset{c}{\hookrightarrow} L^2(0,l)$. Then, it follows that each space from the Cartesian product in the definition of $D(\mathcal{A})$ has compact embedding on $\mathcal{H}$. Therefore, by Proposition 2.1, $\rho(\mathcal{A})$ is also compact. Furthermore by Proposition 2.2, it follows $\sigma(\mathcal{A}) = \mathbb{C} \setminus \rho(\mathcal{A})$ has only eigenvalues from $\mathcal{A}$. Assuming $i\mathbb{R} \subsetneq \rho(\mathcal{A})$, then exists $\lambda \in \mathbb{R}$, such that, $i\lambda \notin \rho(\mathcal{A}) - \{0\}$ (Lemma 4.1). Thus, $i\lambda \in \sigma(\mathcal{A})$, i.e., there is $\mathbf{U} = (\varphi, \Phi, \psi, \Psi) \in D(\mathcal{A})$, such that

$$\mathcal{A}\mathbf{U} - i\lambda\mathbf{U} = 0. \tag{4.11}$$

Taking the inner product in $\mathcal{H}$ of (4.11) by $\mathbf{U}$, we have

$$0 = (\mathcal{A}\mathbf{U} - i\lambda\mathbf{U}, \mathbf{U})_{\mathcal{H}} = (\mathcal{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}} - i\lambda\|\mathbf{U}\|_{\mathcal{H}}^2.$$

By taking the real part and considering (3.14), we have the following

$$0 = \text{Re}(\mathcal{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}} = -b_{11}\|\Phi\|_2^2 - (b_{12} + b_{21})\text{Re}(\Psi, \Phi)_2 - b_{22}\|\Psi\|_2^2.$$

The above and the fact that B is a positive definite matrix imply, through (2.2) from Lemma 2.51 (Note 13), that $\Phi = \Psi = 0$. On the other hand, rewriting (4.11) in terms of its coordinates

$$\Phi - i\lambda\varphi = 0, \tag{4.12}$$
$$\frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{b_{11}}{\rho_1}\Phi - \frac{b_{12}}{\rho_1}\Psi - i\lambda\Phi = 0, \tag{4.13}$$
$$\Psi - i\lambda\psi = 0, \tag{4.14}$$
$$\frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) - \frac{b_{21}}{\rho_2}\Phi - \frac{b_{22}}{\rho_2}\Psi - i\lambda\Psi = 0. \tag{4.15}$$

Replacing $\Phi = \Psi = 0$ in (4.12) and (4.14) we get $\varphi = \psi = 0$. Therefore, $\mathbf{U} = (0,0,0,0)$, which is a contradiction since $\mathbf{U} \neq 0$ is an eigenvector from $\mathcal{A}$. Hence, it follows that $i\mathbb{R} \subset \rho(\mathcal{A})$. □

Lemma 4.3. *Consider* $\rho_1, \rho_2, b, k > 0$ *and let* B *a positive definite matrix. Then,*

$$\lim_{|\lambda|\to\infty} \sup \|(i\lambda I_{\mathcal{H}} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. By the previous Lemma, $i\mathbb{R} \subset \rho(\mathcal{A})$. Therefore, given $\mathbf{F} \in \mathcal{H}$ there is $\mathbf{U} \in D(\mathcal{A})$, such that

$$(i\lambda I_{\mathcal{H}} - \mathcal{A})\mathbf{U} = \mathbf{F}, \ \ \forall \ \lambda \in \mathbb{R}. \tag{4.16}$$

Let $\mathbf{F} = (f_1, f_2, f_3, f_4)^\top \in \mathcal{H}$ and $\mathbf{U} = (\varphi, \Phi, \psi, \Psi)^\top \in D(\mathcal{A})$ satisfying (4.16), such that, $(i\lambda I_{\mathcal{H}} - \mathcal{A})\mathbf{U} = \mathbf{F}$ then $\mathbf{U} = (i\lambda I_{\mathcal{H}} - \mathcal{A})^{-1}\mathbf{F}$. Thereby, in order to show the limit superior of $\|(i\lambda I_{\mathcal{H}} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}$ is finite when $|\lambda| \to \infty$, it is sufficient to show that exists a positive constant C, such that, for all $\mathbf{F} \in \mathcal{H}$

$$\|\mathbf{U}\|_{\mathcal{H}} \leq C\|\mathbf{F}\|_{\mathcal{H}}. \tag{4.17}$$

We have that $i\lambda\mathbf{U} - \mathcal{A}\mathbf{U} = \mathbf{F}$ and this implies $i\lambda\|\mathbf{U}\|_{\mathcal{H}}^2 - (\mathcal{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}} = (\mathbf{F}, \mathbf{U})_{\mathcal{H}}$. Next, taking the real part, we obtain

$$-\text{Re}(\mathcal{A}\mathbf{U}, \mathbf{U})_{\mathcal{H}} = \text{Re}(\mathbf{F}, \mathbf{U})_{\mathcal{H}}.$$

Therefore, from (3.14)

$$b_{11}\|\Phi\|_2^2 + \text{Re}(b_{12} + b_{21})(\Phi, \Psi)_2 + b_{22}\|\Psi\|_2^2 \leq C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}.$$

By Lemma 2.51 (Note 13),

$$\|\Phi\|_2^2 + \|\Psi\|_2^2 \leq C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}, \tag{4.18}$$

Nonetheless, rewriting the resolvent equation (4.16) on terms of its coordinates, we have

$$i\lambda\varphi - \Phi = f_1, \tag{4.19}$$

$$i\lambda\Phi - \frac{k}{\rho_1}(\varphi_x + \psi)_x + \frac{b_{11}}{\rho_1}\Phi + \frac{b_{12}}{\rho_1}\Psi = f_2, \tag{4.20}$$

$$i\lambda\psi - \Psi = f_3, \tag{4.21}$$

$$i\lambda\Psi - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{b_{21}}{\rho_2}\Phi + \frac{b_{22}}{\rho_2}\Psi = f_4. \tag{4.22}$$

Taking the inner product in $L^2(0, l)$ of the resolvent equation (4.22) with $\overline{\psi}$, we obtain

$$\rho_2\int_0^l i\lambda\Psi\overline{\psi}\,dx - b\int_0^l \psi_{xx}\overline{\psi}\,dx + k\int_0^l (\varphi_x + \psi)\overline{\psi}\,dx + \int_0^l (b_{21}\Phi + b_{22}\Psi)\overline{\psi}\,dx = \rho_2\int_0^l f_4\overline{\psi}\,dx.$$

Using integration by parts and (4.21), we find

$$\begin{aligned} b\int_0^l |\psi_x|^2\,dx = \rho_2\int_0^l \Psi\overline{(\Psi + f_3)}\,dx - k\int_0^l (\varphi_x + \psi)(\overline{\Psi + f_3})\,dx \\ - b_{21}\int_0^l \Phi\overline{\psi}\ dx - b_{22}\int_0^l \Psi\overline{\psi}\ dx + \rho_2\int_0^l f_4\overline{\psi}\ dx. \end{aligned}$$

Using Cauchy-Schwarz and Poincaré inequalities, we obtain

$$b\|\psi_x\|_2^2 \le \rho_2\|\Psi\|_2^2 + \rho_2\|\Psi\|_2\|f_3\|_2 + k\|\varphi_x+\psi\|_2\|\Psi\|_2 + k\|\varphi_x+\psi\|_2\|f_3\|_2 \\ + |b_{21}|\|\Phi\|_2\|\psi\|_2 + b_{22}\|\Psi\|_2\|\psi\|_2 + \rho_2\|f_4\|_2\|\psi\|_2.$$

By using (4.18) and Young Ineqaulity

$$b\|\psi_x\|_2^2 \le \epsilon\|\psi_x\|_2^2 + C_\epsilon\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}} + C\|\varphi_x+\psi\|_2\|\Psi\|_2.$$

Taking $\epsilon = \frac{b}{2} > 0$, there exists $C > 0$, such that

$$b\|\psi_x\|_2^2 \le C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2. \tag{4.23}$$

Now, taking the inner product in $L^2(0,l)$ of the resolvent equation (4.20) with $\overline{\varphi}$, we get

$$\rho_1\int_0^l i\lambda\Phi\overline{\varphi}\,dx - k\int_0^l(\varphi_x+\psi)_x\overline{\varphi}\,dx + b_{11}\int_0^l\Phi\overline{\varphi}\,dx + b_{12}\int_0^l\Psi\overline{\varphi}\,dx = \rho_1\int_0^l f_2\overline{\varphi}\,dx.$$

Using integration by parts and (4.19)

$$k\int_0^l|\varphi_x+\psi|\,dx = \rho_1\int_0^l\Phi\overline{(\Phi+f_1)}\;dx + k\int_0^l(\varphi_x+\psi)\overline{\psi}\;dx \\ - b_{11}\int_0^l\Phi\overline{\varphi}\,dx - b_{12}\int_0^l\Psi\overline{\varphi}\,dx + \rho_1\int_0^l f_2\overline{\varphi}\,dx.$$

Using Cauchy-Schwarz and Hölder inequalities, we get

$$k\|\varphi_x+\psi\|_2^2 \le \rho_1\|\Phi\|_2^2 + \rho_1\|\Phi\|_2\|f_1\|_2 + k\|\varphi_x+\psi\|_2\|\psi\|_2 \\ + b_{11}\|\Phi\|_2\|\mathbf{U}\|_{\mathcal{H}} + |b_{12}|\|\Psi\|_2\|\mathbf{U}\|_{\mathcal{H}} + \rho_1\|\varphi\|_2\|f_2\|_2.$$

Using the Young and Poincaré inequalities

$$k\|\varphi_x+\psi\|_2^2 \le C\|\Phi\|_2^2 + \epsilon\|\varphi_x+\psi\|_2^2 + C_\epsilon\|\psi_x\|_2^2 + C\|\Phi\|_2\|\mathbf{U}\|_{\mathcal{H}} \\ + C\|\Psi\|_2\|\mathbf{U}\|_{\mathcal{H}} + C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}.$$

Taking $\epsilon = \frac{k}{2} > 0$ there exists $C > 0$, such that

$$k\|\varphi_x+\psi\|_2^2 \le C\|\psi_x\|_2^2 + C\|\Phi\|_2\|\mathbf{U}\|_{\mathcal{H}} + C\|\Psi\|_2\|\mathbf{U}\|_{\mathcal{H}} + C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}. \tag{4.24}$$

Finally, from (4.18), (4.23) and (4.24), we arrive at

$$\begin{aligned}\|\mathbf{U}\|_{\mathcal{H}}^2 &= \rho_1\|\Phi\|_2^2 + \rho_2\|\Psi\|_2^2 + b\|\varphi_x\|_2^2 + k\|\varphi_x + \psi\|_2^2 \\ &\leq C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}} \\ &\leq \epsilon\|\mathbf{U}\|_{\mathcal{H}}^2 + C_\epsilon\|\mathbf{F}\|_{\mathcal{H}}^2.\end{aligned}$$

Therefore, for $\epsilon < 1$, we find (4.17). The proof is complete. □

Theorem 4.4. *Let* $\rho_1, \rho_2, k, b > 0$. *The Timoshenko system* (3.1)-(3.4) *is exponentially stable if the damping matrix*

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

is a positive definite matrix.

Proof. Through the Lemmas 4.1, 4.2 and 4.3, we conclude the proof of Theorem 4.4 by Theorem 2.47. Consequently, the Timoshenko system is exponentially stable. □

5 EXPONENTIAL STABILITY - A PARTICULAR CASE OF B NON-POSITIVE DEFINITE

5.1 THE PROBLEM

Here, we study the exponential stability for the following Timoshenko system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \ \text{ in } \ (0,l) \times (0,\infty), \tag{5.1}$$
$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \psi_t = 0 \ \text{ in } \ (0,l) \times (0,\infty), \tag{5.2}$$

with initial conditions

$$\varphi(\cdot,0) = \varphi_0(\cdot),\ \varphi_t(\cdot,0) = \varphi_1(\cdot),\ \psi(\cdot,0) = \psi_0(\cdot),\ \psi_t(\cdot,0) = \psi_1(\cdot), \tag{5.3}$$

and Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(l,t) = \psi(0,t) = \psi(l,t) = 0, \ \ t \geq 0. \tag{5.4}$$

5.2 EXISTENCE AND UNIQUENESS

Before presenting the existence and uniqueness result, let us compile some information from Chapter 3.

1. The problem (5.1)- (5.4) can be expressed abstractly as the following Cauchy problem

$$\begin{aligned} \mathbf{U}_t &= \mathcal{A}\mathbf{U},\ t > 0, \\ \mathbf{U}(0) &= \mathbf{U}_0, \end{aligned} \tag{5.5}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined in (3.5) with specific coefficients $b_{11} = b_{12} = b_{21} = 0$ and $b_{22} = 1$. The phase space $\mathcal{H}$ is described in (3.7), and the domain of the operator $\mathcal{A}$ is established by Proposition 3.1.

2. The existence and uniqueness result can also be derived from Theorem 3.4 in this scenario, where the matrix B satisfies Condition 01 as given in (3.12) (see Example 4, Section 3.3). Specifically, we obtain the following result.

Theorem 5.1 (Existence and Uniqueness)**.** *If* $\mathbf{U}_0 \in D(\mathcal{A})$*, then the Cauchy Problem* (5.5) *has a unique solution* $\mathbf{U} \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H})$.

5.3 EXPONENTIAL STABILITY FOR EQUAL WAVE SPEEDS

From now on, we will study the exponential stability of the system (5.1)-(5.4) using Prüss Theorem (Theorem 2.47). In order to keep up with our proof, we shall define the resolvent equation for the system (5.1)-(5.2). The resolvent equation is given by

$$i\lambda\mathbf{U} - \mathcal{A}\mathbf{U} = \mathbf{F}. \tag{5.6}$$

Also, rewriting (5.6) in terms of coordinates we have:

$$i\lambda\varphi - \Phi = f_1, \tag{5.7}$$

$$i\lambda\Phi - \frac{k}{\rho_1}(\varphi_x + \psi)_x = f_2, \tag{5.8}$$

$$i\lambda\psi - \Psi = f_3, \tag{5.9}$$

$$i\lambda\Psi - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{\Psi}{\rho_2} = f_4. \tag{5.10}$$

Lemma 5.2. *Let* $\rho_1, \rho_2, b, k > 0$*. Then,* $0 \in \rho(\mathcal{A})$*.*

Proof. To demonstrate that $0 \in \rho(\mathcal{A})$ it is sufficient to show that $(-\mathcal{A})^{-1}$ exists and it is bounded. Given $\mathbf{F} \in \mathcal{H}$, we will show that

$$-\mathcal{A}\mathbf{U} = \mathbf{F} \tag{5.11}$$

has a unique solution $\mathbf{U} \in D(\mathcal{A})$. Rewriting (5.11) in terms of its components, we have

$$-\Phi = f_1, \tag{5.12}$$

$$-k(\varphi_x + \psi)_x = \rho_1 f_2, \tag{5.13}$$

$$-\Psi = f_3, \tag{5.14}$$

$$-b\psi_{xx} + k(\varphi_x + \psi) - \Psi = \rho_2 f_4. \tag{5.15}$$

From (5.14), $\Psi = -f_3$. Replacing in (5.15) we find

$$-b\psi_{xx} + k(\varphi_x + \psi) = \rho_2 f_4 - f_3.$$

Now, defining

$$\begin{aligned} g_1 &\equiv \rho_1 f_2, \\ g_2 &\equiv \rho_2 f_4 - f_3, \end{aligned}$$

we obtain the following system

$$\begin{aligned} -k(\varphi_x + \psi)_x &= g_1, \\ -b\psi_{xx} + k(\varphi_x + \psi) &= g_2, \end{aligned}$$

which has a solution, as we can see in Theorem 3.4. Therefore, $(-\mathcal{A})^{-1}$ exists. Now, we shall demonstrate that $-\mathcal{A}^{-1}$ is bounded, to this end we just need to show that there exists a positive constant C, such that

$$\|\mathbf{U}\|_{\mathcal{H}} \le C\|\mathbf{F}\|_{\mathcal{H}}.$$

Using (5.7) and by Poincaré inequality, we get

$$\begin{aligned}\|\Phi\|_2^2 = \|f_1\|_2^2 \le c_p^2\|f_{1,x}\|_2^2 &\le c_p^2(\|f_{1,x} + f_3\|_2 + \|f_3\|_2)^2 \\ &\le 2c_p^2\|f_{1,x} + f_3\|_2^2 + 2c_p^4\|f_{3,x}\|_2^2 \\ &\le C\|\mathbf{F}\|_{\mathcal{H}}.\end{aligned}$$

Similarly, from (5.9), we have

$$\|\Psi\|_2^2 = \|f_3\|_2^2 \le c_p^2\|f_{3,x}\|_2^2 \le C\|\mathbf{F}\|_{\mathcal{H}}.$$

Taking the inner product of (5.13) with $\overline{\varphi}$ in $L^2(0,l)$, we have

$$-k\int_0^l (\varphi_x + \psi)_x\overline{\varphi}\,dx = \rho_1\int_0^l f_2\overline{\varphi}\,dx,$$

integrating by parts

$$k\int_0^l (\varphi_x + \psi)(\overline{\varphi_x + \psi})\,dx - k\int_0^l (\varphi_x + \psi)\overline{\psi}\,dx = \rho_1\int_0^l f_2\overline{\varphi}\,dx$$

from where we get

$$k\|\varphi_x + \psi\|_2^2 - k(\varphi_x + \psi, \psi)_2 = \rho_1(f_2, \varphi)_2. \tag{5.16}$$

Also, taking the inner product of (5.15) with $\overline{\psi}$ in $L^2(0,l)$, we have

$$-b\int_0^l \psi_{xx}\overline{\psi}\,dx + k\int_0^l (\varphi_x + \psi)\overline{\psi}\,dx - \int_0^l \Psi\overline{\psi}\,dx = \rho_2\int_0^l f_4\overline{\psi}\,dx,$$

integrating by parts

$$b\int_0^l \psi_x\overline{\psi_x}\,dx + k\int_0^l (\varphi_x + \psi)\overline{\psi}\,dx - \int_0^l \Psi\overline{\psi}\,dx = \rho_2\int_0^l f_4\overline{\psi}\,dx,$$

then

$$b\|\varphi_x\|_2^2 + k(\varphi_x + \psi, \psi)_2 - (\Psi, \psi)_2 = \rho_2(f_4, \psi)_2. \tag{5.17}$$

Adding (5.16) and (5.17) we have

$$k\|\varphi_x + \psi\|_2^2 + b\|\varphi_x\|_2^2 = (\Psi, \psi)_2 + \rho_1(f_2, \varphi)_2 + \rho_2(f_4, \psi)_2.$$

Using Cauchy-Schwarz and Poincaré inequalities

$$\begin{aligned}
&k\|\varphi_x+\psi\|_2^2+b\|\varphi_x\|_2^2\\
&\qquad\le c_p\rho_1\|f_2\|_2\|\varphi_x\|_2+c_p\rho_2\|f_4\|_2\|\psi_x\|_2+c_p\|\Psi\|_2\|\psi_x\|_2\\
&\qquad\le c_p\rho_1\|f_2\|_2\big(\|\varphi_x+\psi\|_2+\|\psi\|_2\big)\;+\;c_p\rho_2\|f_4\|_2\|\psi_x\|_2+c_p\|\Psi\|_2\|\psi_x\|_2\\
&\qquad\le c_p\rho_1\|f_1\|_2\|\varphi_x+\psi\|_2+c_p^2\rho_1\|f_1\|_2\|\psi_x\|_2+c_p\rho_2\|f_4\|_2\|\psi_x\|_2+c_p\|\Psi\|_2\|\psi_x\|_2\\
&\qquad\le C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}+C\|\Psi\|_2\|\mathbf{U}\|_2,
\end{aligned}$$

for some $C>0$. Applying the Young inequality, yields

$$\|\mathbf{U}\|_{\mathcal{H}}\le C\|\mathbf{F}\|_{\mathcal{H}}^2.$$

Hence, there is $C>0$ such that

$$\|(-\mathcal{A})^{-1}\mathbf{F}\|_{\mathcal{H}}\le C\|\mathbf{F}\|_{\mathcal{H}},\ \ \forall\ \mathbf{F}\in\mathcal{H}.$$

Hence, the proof is complete. □

Lemma 5.3. *Suppose* $\rho_1,\rho_2,b,k>0$. *Then,* $i\mathbb{R}\subset\rho(\mathcal{A})$.

Proof. By Theorem 2.41, it follows that $D(\mathcal{A})\overset{c}{\hookrightarrow}\mathcal{H}$. By the Proposition 2.2, $\sigma(\mathcal{A})=\mathbb{C}\setminus\rho(\mathcal{A})$ has only eigenvectors of $\mathcal{A}$. If we assume that $i\mathbb{R}\subsetneq\rho(\mathcal{A})$ then exists $i\lambda\notin\rho(\mathcal{A})-\{0\}$ (Lemma 5.2). Thus, $i\lambda\in\sigma(\mathcal{A})$, i.e., exists $\mathbf{U}=(\varphi,\Phi,\psi,\Psi)^\top\neq 0$, such that

$$i\lambda\mathbf{U}-\mathcal{A}\mathbf{U}=0. \tag{5.18}$$

Taking the inner product of (5.18), we obtain

$$0=\mathrm{Re}\Big\{i\lambda\|\mathbf{U}\|_2^2-(\mathcal{A}\mathbf{U},\mathbf{U})_{\mathcal{H}}\Big\}=-\mathrm{Re}(\mathcal{A}\mathbf{U},\mathbf{U})=\|\Psi\|_2^2.$$

Therefore, $\Psi=0$. Replacing $\Psi=0$ into (5.9) (with $\mathbf{F}=(0,0,0,0)$), we have that, $\psi=0$ and, from (5.10) (also with $\mathbf{F}=(0,0,0,0)$) we have $\varphi=0$ and from where we get, in (5.7), $\Phi=0$. Therefore $\mathbf{U}=(0,0,0,0)$ which is an contraction, because $\mathbf{U}$ is an eigenvector of $\mathcal{A}$. Hence, $i\mathbb{R}\subset\rho(\mathcal{A})$. □

Lemma 5.4. *Let* $\rho_1,\rho_2,b,k>0$. *Then,*

$$\|\Psi\|_2^2\le\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}.$$

Proof. Taking the inner product of equation (5.6) with $\mathbf{U}$, we find

$$\mathrm{Re}(\mathbf{F},\mathbf{U})_{\mathcal{H}}=\mathrm{Re}\Big\{i\lambda\|\mathbf{U}\|_2^2-(\mathcal{A}\mathbf{U},\mathbf{U})_{\mathcal{H}}\Big\}=-\mathrm{Re}(\mathcal{A}\mathbf{U},\mathbf{U})=\|\Psi\|_2^2.$$

The conclusion follows from Cauchy-Schwarz inequality. □

Lemma 5.5. *Let $\rho_1, \rho_2, b, k > 0$. There is $C > 0$ such that*

$$b\|\psi_x\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}.$$

Proof. Taking the inner product in $L^2(0,l)$ of the equation (5.10) with $\overline{\psi}$, we have

$$i\lambda\rho_2\int_0^l \Psi\overline{\psi}\,dx - b\int_0^l \psi_{xx}\overline{\psi}\,dx + k\int_0^l(\varphi_x+\psi)\overline{\psi}\,dx + \int_0^l \Psi\overline{\psi}\,dx = \rho_2\int_0^l f_4\overline{\psi}\,dx.$$

Using (5.9)

$$-\rho_2\int_0^l \Psi\overline{(\Psi+f_3)}\,dx - b\|\psi_x\|_2^2 - \frac{k}{i\lambda}\int_0^l(\varphi_x+\psi)\overline{(\Psi+f_3)}\,dx + \int_0^l \Psi\overline{\psi}\,dx = \rho_2\int_0^l f_4\overline{\psi}\,dx.$$

Using Cauchy-Schwarz, Poincaré and Young inequalities, we have

$$b\|\psi_x\|_2^2 \leq \rho_2\|\Psi\|_2\big(\|\Psi\|_2+\|f_3\|_2\big)+\frac{k}{|\lambda|}\|\varphi_x+\psi\|_2\big(\|\Psi\|_2+\|f_3\|_2\big)+C\|\psi\|_2\big(\|\Psi\|_2+\|f_4\|_2\big).$$

Therefore,

$$b\|\psi_x\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 + C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}.$$

Using Lemma 5.4

$$b\|\psi_x\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}.$$

Hence, the proof is complete. □

Lemma 5.6. *Let $\rho_1, \rho_2, b, k > 0$ and $\frac{k}{b} = \frac{\rho_1}{\rho_2}$. Then, there is $C > 0$ such that*

$$k\|\varphi_x+\psi\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 - \mathrm{Re}\left\{ b\overline{\varphi_x}\psi_x\Big|_{x=0}^{x=l}\right\}.$$

Proof. Now, taking the inner product of (5.10) with $\overline{(\varphi_x+\psi)}$

$$\begin{aligned} k\|\varphi_x+\psi\|_2^2 &= -i\lambda\rho_2\int_0^l \Psi\overline{(\varphi_x+\psi)}\,dx + \underbrace{b\int_0^l \psi_{xx}\overline{(\varphi_x+\psi)}\,dx}_{:=R_1} \\ &\quad - \int_0^l \Psi\overline{\varphi_x}\,dx - \int_0^l \Psi\overline{\psi}\,dx + \rho_2\int_0^l f_4\overline{(\varphi_x+\psi)}\,dx. \end{aligned} \tag{5.19}$$

Integrating R_1 by parts

$$b\int_0^l \psi_{xx}\overline{(\varphi_x+\psi)}\,dx = b\overline{(\varphi_x+\psi)}\psi_x\Big|_{x=0}^{x=l} - b\int_0^l \psi_x\overline{(\varphi_x+\psi)_x}\,dx. \tag{5.20}$$

Replacing (5.7), (5.9) and (5.20) into (5.19), we have

$$\begin{aligned} k\|\varphi_x+\psi\|_2^2 &= \rho_2\int_0^l \Psi\overline{\Phi_x}\,dx+\rho_2\int_0^l \Psi\overline{f_{1,x}}\,dx+\rho_2\|\Psi\|_2^2+\rho_2\int_0^l \Psi\overline{f_3}\,dx \\ &\quad +b\overline{(\varphi_x+\psi)}\psi_x\Big|_{x=0}^{x=l}-\underbrace{b\int_0^l \psi_x\overline{(\varphi_x+\psi)_x}\,dx}_{:=R_2}-\int_0^l \Psi\overline{\varphi_x}\,dx \\ &\quad -\int_0^l \Psi\overline{\psi}\,dx+\rho_2\int_0^l f_4\overline{(\varphi_x+\psi)}\,dx. \end{aligned} \tag{5.21}$$

Now, replacing $(\varphi_x+\psi)_x$ given on (5.8) at R_2, we obtain

$$\begin{aligned} -b\int_0^l \psi_x\overline{(\varphi_x+\psi)_x}\,dx &= -\frac{b\rho_1}{k}\int_0^l \psi_x\overline{(i\lambda\rho_1\Phi-f_2)}\,dx \\ &= -i\lambda\frac{b\rho_1}{k}\int_0^l \psi\overline{\Phi_x}\,dx+\frac{b\rho_1}{k}\int_0^l \psi_x\overline{f_2}\,dx. \end{aligned}$$

Replacing, again, ψ given on (5.9)

$$-i\lambda\frac{b\rho_1}{k}\int_0^l \psi\overline{\Phi_x}\,dx+\frac{b\rho_1}{k}\int_0^l \psi_x\overline{f_2}\,dx=-\frac{b\rho_1}{k}\int_0^l (\Psi+f_3)\overline{\Phi_x}\,dx+\frac{b\rho_1}{k}\int_0^l \psi_x\overline{f_2}\,dx.$$

Returning to (5.21) we get

$$\begin{aligned} k\|\varphi_x+\psi\|_2^2 &= \left(\rho_2-\frac{b\rho_1}{k}\right)\int_0^l \Psi\overline{\Phi_x}\,dx+\rho_2\int_0^l \Psi\overline{f_{1,x}}\,dx+\rho_2\|\Psi\|_2^2+\rho_2\int_0^l \Psi\overline{f_3}\,dx \\ &\quad +b\overline{(\varphi_x+\psi)}\psi_x\Big|_{x=0}^{x=l}+\frac{b\rho_1}{k}\int_0^l f_{3,x}\overline{\Phi}\,dx+\frac{b\rho_1}{k}\int_0^l \psi_x\overline{f_2}\,dx \\ &\quad -\int_0^l \Psi\overline{\varphi_x}\,dx-\int_0^l \Psi\overline{\psi}\,dx+\rho_2\int_0^l f_4\overline{\varphi_x}\,dx+\rho_2\int_0^l f_4\overline{\psi}\,dx. \end{aligned}$$

Note that, from the hypothesis, it follows that

$$\frac{\rho_1}{\rho_2}-\frac{k}{b} \text{ if and only if } \rho_2=\frac{b\rho_1}{k}.$$

Then,

$$\left(\rho_2-\frac{b\rho_1}{k}\right)\int_0^l \Psi\overline{\Phi_x}\,dx=0.$$

Therefore, by the Cauchy-Schwarz Ineqaulity, we have

$$\begin{aligned} k\|\varphi_x+\psi\|_2^2 \;\leq\; & \rho_2\|\Psi\|_2\|f_{1,x}\|_2+\rho_2\|\Psi\|_2^2+\rho_2\|\Psi\|_2\|f_3\|_2+\frac{b\rho_1}{k}\|f_{3,x}\|_2\|\Phi\|_2 \\ & +\frac{b\rho_1}{k}\|\psi_x\|_2\|f_2\|_2 \;+\; \|\Psi\|_2\|\varphi_x+\psi\|_2+\rho_2\|f_4\|_2\|\varphi_x \;+\; \psi\|_2 \\ & +\text{Re}\left\{ b\overline{(\varphi_x+\psi)}\psi_x\Bigg|_{x=0}^{x=l}\right\}. \end{aligned}$$

The above implies that

$$k\|\varphi_x+\psi\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2+\text{Re}\left\{ b\overline{\varphi_x}\psi_x\Bigg|_{x=0}^{x=l}\right\},$$

which concludes the proof.

□

Lemma 5.7. *Let* $\rho_1,\rho_2,b,k>0$, $b_{ij}\in\mathbb{R}$ *and* $\xi\in C^1([0,l])$, *such that,* $\xi(0)=-\xi(l)=1$. *There is* $C>0$ *such that*

(i)

$$-\frac{b}{2}\xi(x)|\psi_x|^2\Bigg|_{x=0}^{x=l} \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+C\|\Psi\|_2\|\mathbf{U}\|_{\mathcal{H}}+C\|\psi_x\|_2^2+C\|\varphi_x+\psi\|_2\|\psi_x\|_2.$$

(ii)

$$-\frac{k}{2}\xi(x)|\varphi_x|^2\Bigg|_{x=0}^{x=l} \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2+C\|\mathbf{U}\|_2^2.$$

Proof. **Proof of (i).** Taking the inner product of (5.10) by $\xi\overline{\psi_x}$ in $L_2(0,l)$, we deduce

$$\underbrace{\rho_2 i\lambda\int_0^l\xi\Psi\overline{\psi_x}\,dx}_{:=R_3}\underbrace{-b\int_0^l\xi\psi_{xx}\overline{\psi_x}dx}_{:=R_4}+k\int_0^l\xi(\varphi_x+\psi)\overline{\psi_x}dx+\int_0^l\Psi\psi_x\xi dx=\rho_2\int_0^l\xi f_4\overline{\psi_x}dx. \quad (5.22)$$

Replacing ψ given by (5.9) in R_3, we obtain

$$\begin{aligned} \text{Re}\{R_3\} &= \text{Re}\left\{-\rho_2\int_0^l\xi\Psi\overline{\Psi_x}\,dx-\rho_2\int_0^l\xi\Psi\overline{f_{3,x}}\,dx\right\} \\ &= \text{Re}\left\{\frac{-\rho_2}{2}\int_0^l\xi\frac{d}{dx}|\Psi|^2\,dx-\rho_2\int_0^l\xi\Psi\overline{f_{3,x}}\,dx\right\} \\ &= \frac{\rho_2}{2}\int_0^l\xi'|\Psi|^2\,dx-\text{Re}\left\{\rho_2\int_0^l\xi\Psi\overline{f_{3,x}}\,dx\right\}. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\text{Re}\{R_4\} &= -\text{Re}\left\{\frac{b}{2}\int_0^l \xi\frac{d}{dx}|\psi_x|^2\,dx\right\}\\ &= -\frac{b}{2}\xi(x)|\psi_x|^2\Bigg|_{x=0}^{x=l} + \frac{b}{2}\int_0^l \xi'|\psi_x|^2\,dx.\end{aligned}$$

Replacing in (5.22), we get

$$\begin{aligned}-\frac{b}{2}\xi(x)|\psi_x|^2\Bigg|_{x=0}^{x=l} =\ & \text{Re}\left\{\rho_2\int_0^l \xi f_4\overline{\psi_x}\,dx\ +\frac{\rho_2}{2}\int_0^l \xi'|\Psi|^2\,dx\ +\ \rho_2\int_0^l \xi\Psi\overline{f_{3,x}}\,dx\right\}\\ & - \text{Re}\left\{\frac{b}{2}\int_0^l \xi'|\psi_x|^2\,dx + k\int_0^l \xi(\varphi_x+\psi)\overline{\psi_x}\,dx + \int_0^l \xi\Psi\overline{\psi_x}\,dx\right\}.\end{aligned} \tag{5.23}$$

We shall estimate the right-hand side of equation (5.23). Here, we handle the first integral, and the other we use the same argument. By Hölder inequality, we find

$$\begin{aligned}\text{Re}\left\{\rho_2\int_0^l \xi f_4\overline{\psi_x}\,dx\right\} &= \left|\rho_2\int_0^l \xi f_4\overline{\psi_x}\,dx\right|\\ &\le \rho_2\|\xi\|\|f_4\|_2\|\psi_x\|_2\\ &\le C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}.\end{aligned}$$

where $\|\xi\| = \sup\{\xi(x)|x\in[0,l]\}$. Therefore

$$-\frac{b^2}{2}\xi(x)|\psi_x|^2\Bigg|_{x=0}^{x=l} \le C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\Psi\|_2\|\mathbf{U}\|_{\mathcal{H}} + C\|\psi_x\|_2^2 + C\|\varphi_x+\psi\|_2\|\psi_x\|_2.$$

This concludes the proof of (i).

Proof of (ii). Taking the inner product of (5.8) by $\xi\overline{\varphi_x}$ in $L^2(0,l)$, we obtain

$$\rho_1 i\lambda\int_0^l \xi\Phi\overline{\varphi_x}\,dx - k\int_0^l \xi(\varphi_x+\psi)_x\overline{\varphi_x}\,dx = \rho_1\int_0^l \xi f_2\overline{\varphi_x}\,dx.$$

Replacing φ given by (5.7)

$$-\rho_1\int_0^l \xi\Phi\overline{f_{1,x}}\,dx \underbrace{-\rho_1\int_0^l \xi\Phi\overline{\Phi}_x\,dx}_{:=R_5}\underbrace{-k\int_0^l \xi\varphi_{xx}\overline{\varphi_x}\,dx}_{:=R_6} -k\int_0^l \xi\psi_x\overline{\varphi_x}\,dx = \rho_1\int_0^l \xi f_2\overline{\varphi_x}\,dx. \tag{5.24}$$

Furthermore, integrating by parts, we obtain

$$\begin{aligned}\text{Re}\{R_5\} &= -\text{Re}\left\{\frac{\rho_1}{2}\int_0^l \xi \frac{d}{dx}|\Phi|^2\,dx\right\}\\ &= \frac{\rho_1}{2}\int_0^l \xi'|\Phi|^2\,dx\end{aligned}$$

and

$$\begin{aligned}\text{Re}\{R_6\} &= -\text{Re}\left\{\frac{k}{2}\int_0^l \xi(x)\frac{d}{dx}|\varphi_x|^2\;dx\right\}\\ &= -\frac{k}{2}\xi(x)|\varphi_x|^2\Big|_{x=0}^{x=l} + \frac{k}{2}\int_0^l \xi|\varphi_x|^2\;dx.\end{aligned}$$

Replacing in (5.24), we get

$$\begin{aligned}-\frac{k}{2}\xi(x)|\varphi_x|^2\Big|_{x=0}^{x=l} &= \text{Re}\left\{\rho_1\int_0^l \xi f_2\overline{\varphi_x}\;dx + \rho_1\int_0^l \xi\Phi\overline{f_{1,x}}\;dx - \rho_1\int_0^l \xi'|\Phi|^2\;dx\right\}\\ &\quad+\text{Re}\left\{-\frac{k}{2}\int_0^l \xi'|\varphi_x|^2\;dx + k\int_0^l \xi\psi_x\overline{\varphi_x}\;dx\right\}. \qquad (5.25)\end{aligned}$$

By Hölder, triangle and Poincaré inequalities, we have

$$\begin{aligned}\text{Re}\left\{-\frac{k}{2}\int_0^l \xi|\varphi_x|^2\;dx\right\} &\le \frac{k}{2}\int_0^l |\xi||\varphi_x|^2\;dx\\ &\le \frac{k}{2}\int_0^l |\xi||\varphi_x+\psi|^2\;dx + \frac{k}{2}\int_0^l |\xi||\psi|^2\;dx\\ &\le C\|\varphi_x+\psi\|_2^2 + C\|\psi_x\|_2^2\\ &\le C\|\mathbf{U}\|_{\mathcal{H}}^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\text{Re}\left\{k\int_0^l \xi\psi_x\overline{\varphi_x}\;dx\right\} &\le k\int_0^l |\xi||\psi_x||\varphi_x|\;dx\\ &\le k\int_0^l |\xi||\psi_x||\varphi_x+\psi|\;dx + k\int_0^l |\xi||\psi_x||\psi|\;dx\\ &\le C\|\psi_x\|_2\|\varphi_x+\psi\|_2 + C\|\psi_x\|_2^2\\ &\le C\|\mathbf{U}\|_{\mathcal{H}}^2.\end{aligned}$$

Replacing in (5.25), we have

$$-\frac{k}{2}\xi(x)|\varphi_x|^2\Big|_{x=0}^{x=l} \le C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 + C\|\mathbf{U}\|_2^2.$$

The proof is complete. □

Corollary 5.1. *Let $\rho_1, \rho_2, b, k > 0$ and $\frac{k}{b} = \frac{\rho_1}{\rho_2}$. Then, there is $C > 0$ such that*

$$k\|\varphi_x + \psi\|_2^2 \leq \epsilon\|\mathbf{U}\|_2^2 + C_\epsilon\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C_\epsilon\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 + C_\epsilon\|\varphi_x + \psi\|_2\|\psi_x\|_2 + C_\epsilon\|\psi_x\|_2^2.$$

Proof. Using Young inequality

$$\mathrm{Re}\left\{ b\overline{\varphi_x}\psi_x \Big|_{x=0}^{x=l} \right\} \leq \epsilon\Big(\xi(x)|\varphi_x|^2\Big)\Big|_{x=0}^{x=l} + C_\epsilon\Big(\xi(x)|\psi_x|^2\Big)\Big|_{x=0}^{x=l}.$$

The conclusion follows from Lemma 5.6 and 5.7 with $\xi(x) = -\frac{2x}{l} + 1$.

□

Lemma 5.8. *Let $\rho_1, \rho_2, b, k > 0$. There is $C > 0$ such that*

$$\rho_1\|\Phi\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 + C\|\varphi_x + \psi\|_2^2 + C\|\psi_x\|_2^2.$$

Proof. Taking the inner product of (5.8) with φ in $L^2(0,l)$, we obtain

$$i\lambda\rho_1\int_0^l \Phi\overline{\varphi}\ dx - k\int_0^l (\varphi_x + \psi)_x\overline{\varphi}\ dx = \rho_1\int_0^l f_2\overline{\varphi}\ dx.$$

Replacing φ given by (5.7), we have

$$\rho_1\|\Phi\|_2^2 = -k\int_0^l (\varphi_x + \psi)_x\overline{\varphi}\ dx - \rho_1\int_0^l \Phi\overline{f_1}\ dx - \rho_1\int_0^l f_2\overline{\varphi}\ dx.$$

Note that, applying the Cauchy-Schwarz, triangle and Poincaré inequalities, we obtain

$$\begin{aligned}
\left|k\int_0^l (\varphi_x + \psi)_x\overline{\varphi}\ dx\right| &= \left|k\int_0^l (\varphi_x + \psi)\overline{\varphi_x}\ dx\right| \\
&= \left|k\int_0^l (\varphi_x + \psi)^2\ dx + k\int_0^l (\varphi_x + \psi)\overline{\psi}\ dx\right| \\
&\leq k\|\varphi_x + \psi\|_2^2 - k\|\varphi_x + \psi\|_2\|\psi\|_2 \\
&\leq C\|\varphi_x + \psi\|_2^2 + C\|\psi_x\|_2^2.
\end{aligned}$$

Therefore

$$\rho_1\|\Phi\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 + C\|\varphi_x + \psi\|_2^2 + C\|\psi_x\|_2^2.$$

Which completes the proof. □

Lemma 5.9. *Consider* $\rho_1, \rho_2, b, k > 0$ *and* $\frac{k}{b} = \frac{\rho_1}{\rho_2}$. *Then,*

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I_{\mathcal{H}} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{5.26}$$

Proof. Once $i\mathbb{R} \subset \rho(\mathcal{A})$, given $\mathbf{F} \in \mathcal{H}$, there is a unique $\mathbf{U} \in D(\mathcal{A})$ such that

$$(i\lambda I_{\mathcal{H}} - \mathcal{A})\mathbf{U} = \mathbf{F}, \ \ \forall \lambda \in \mathbb{R},$$

then

$$\mathbf{U} = (i\lambda I_{\mathcal{H}} - \mathcal{A})^{-1}.$$

In order to show that (5.26), we just need to demonstrate that there is a constant $C > 0$, such that, for all $\mathbf{F} \in \mathcal{H}$

$$\|\mathbf{U}\|_{\mathcal{H}} \leq C\|\mathbf{F}\|_{\mathcal{H}}.$$

Using Lemma 5.8, we find

$$\rho_1\|\Phi\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 + C\|\varphi_x + \psi\|_2^2 + C\|\psi_x\|_2^2.$$

Also adding $k\|\varphi_x + \psi\|_2^2$ to the inequality above and by Corollary 5.1, we get

$$\begin{aligned} \rho_1\|\Phi\|_2^2 + k\|\varphi_x + \psi\|_2^2 \leq \epsilon\|\mathbf{U}\|_2^2 + C_\epsilon\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C_\epsilon\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 \\ + C_\epsilon\|\varphi_x + \psi\|_2\|\psi_x\|_2 + C_\epsilon\|\psi_x\|_2^2. \end{aligned} \tag{5.27}$$

Now, adding $\rho_2\|\Psi\|_2^2$ and $b\|\psi_x\|_2^2$ into the inequality (5.27), we obtain

$$\|\mathbf{U}\|_2^2 \leq \epsilon\|\mathbf{U}\|_2^2 + C_\epsilon\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C_\epsilon\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 + C_\epsilon\|\mathbf{U}\|_{\mathcal{H}}\|\psi_x\|_2 + C_\epsilon\|\psi_x\|_2^2. \tag{5.28}$$

Taking $\epsilon = \frac{1}{2}$ in (5.28), we obtain

$$\|\mathbf{U}\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 + C\|\mathbf{U}\|_{\mathcal{H}}\|\psi_x\|_2 + C\|\psi_x\|_2^2. \tag{5.29}$$

Using Young inequality in (5.29)

$$\|\mathbf{U}\|_2^2 \leq \epsilon\|\mathbf{U}\|_2^2 + C_\epsilon\|\Psi\|_2^2 + C_\epsilon\|\psi_x\|_2^2 + C_\epsilon\|\mathbf{F}\|_{\mathcal{H}}^2. \tag{5.30}$$

Taking $\epsilon = \frac{1}{2}$ in (5.30), we find

$$\|\mathbf{U}\|_2^2 \leq C\|\Psi\|_2^2 + C\|\psi_x\|_2^2 + C\|\mathbf{F}\|_{\mathcal{H}}^2.$$

Using Lemma 5.4, we have that

$$\|\mathbf{U}\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{U}\|_{\mathcal{H}}\|\Psi\|_2 + C\|\mathbf{F}\|_{\mathcal{H}}^2. \tag{5.31}$$

Using Young inequality in (5.31)

$$\|\mathbf{U}\|_2^2 \leq \epsilon\|\mathbf{U}\|_{\mathcal{H}}^2 + C_\epsilon\|\Psi\|_2^2 + C_\epsilon\|\mathbf{F}\|_{\mathcal{H}}^2. \tag{5.32}$$

Taking $\epsilon = \frac{1}{2}$ in (5.32) and using Lemma 5.5, we find

$$\|\mathbf{U}\|_2^2 \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{F}\|_{\mathcal{H}}^2. \tag{5.33}$$

Again, Using Young inequality in (5.33) with $\epsilon = \frac{1}{2}$, we find

$$\|\mathbf{U}\|_2^2 \leq \frac{1}{2}\|\mathbf{U}\|_{\mathcal{H}}^2 + C\|\mathbf{F}\|_{\mathcal{H}}^2.$$

Therefore

$$\|\mathbf{U}\|_{\mathcal{H}} \leq C\|\mathbf{F}\|_{\mathcal{H}}.$$

Hence, the proof is complete.

□

Theorem 5.10. *Let $\rho_1, \rho_2, k, b > 0$. The Timoshenko system, given by* (5.1)-(5.4) *is exponentially stable if*

$$\frac{k}{b} = \frac{\rho_1}{\rho_2}.$$

Proof. Through Lemmas 5.2, 5.3, 5.9 we can conclude the proof of Theorem 5.10 by Theorem 2.47. Consequently, the Timoshenko system (5.1)-(5.4) is exponentially stable. □

6 CONCLUSION

In this work, the Timoshenko System was approached in order to extend the findings of [12], motivated by [3], with an specific condition for the damping matrix B, which must be positive definite.

In Chapter 3, we focused on formulating the problem and constructing the semigroup, from where we were able to rigorously establish the system's well-posedness.

Moving forward to chapter 4, our analysis demonstrated the system's exponential stability, particularly when considering the damping matrix B as a positive definite matrix. This finding underscores the importance of proper damping considerations.

Lastly, in Chapter 5, we have studied a particular case of a Timoshenko System, where we investigates a scenario with a constant damping parameter $b_{22} = 1$. Despite the absence of B's positive definiteness, our analysis revealed that stability can still be maintained under certain conditions, showcasing the nuanced interplay between damping parameters and system stability.

In essence, this dissertation contributes to the body of knowledge surrounding the Timoshenko system by providing a comprehensive examination of the role of matrices, well-posedness, and stability considerations. By extending the findings of prior research and unveiling new insights, this work advances our understanding of dynamic systems and lays the groundwork for future explorations in this field.

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