Universidade Estadual de Londrina

GIOVANNA PIMENTA

ASPECTS OF HIGHER-FORM SYMMETRIES

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Dissertação apresentada ao Programa de Mestrado em Física da Universidade Estadual de Londrina para obtenção do título de Mestre em Física

Orientador: Prof. Dr. Pedro Rogério Sergi Gomes

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Londrina, 06 de março de 2023.

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Meu último ano da graduação e primeiro ano do mestrado ocorreram durante o período pandêmico. Assim como todo o mundo, minha vida parou por um período. Meus estudos sofreram um grande golpe. O rendimento foi diferente, a beleza da física ficou diferente, alguns sonhos se foram e outros surgiram no lugar. Mas com o retorno da vida acadêmica em 2022 o amor pela física se reacendeu. Aos poucos tudo voltou a fazer sentido e me vi novamente no lugar certo trabalhando com a coisa certa. Essa dissertação é fruto de todos os altos e baixos desse período.

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One day, the world stopped Without any warning Spring didn't know to wait Showed up not even a minute late Like an echo in the forest The day will come back around As if nothing happened Yeah, life goes on Life Goes On - BTS

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RESUMO

Com o advento do significado topológico para cargas conservadas na TQC, diferentes tipos de simetrias surgiram, essas são as simetrias generalizadas. Neste trabalho vamos focar em um tipo delas, a simetria de higher-form. Esta consiste em agir com a simetria em objetos extendidos. Por exemplo na eletrodinâmica quântica a simetria de 1-forma elétrica age na linha de Wilson. Essa nova simetria também possui um mecanismo de quebra espontânea, o que tem como consequência o fóton como um bóson de Goldstone.

Palavras-chave: Simetrias generalizadas, formas diferenciais, cargas topológicas

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ABSTRACT

With the advent of the topological meaning for the conserved charges in QFT different kinds of symmetries appeared, these are the Generalized Global Symmetries. In this work we will focus on one kind of them, the Higher-Form symmetry. It consists on acting with the symmetry on extended objects. For example in QED the $1-$form electric symmetry acts on the Wilson line. This new symmetry also has a spontaneous symmetry breaking mechanism, and as a consequence leave the photon as the Goldstone boson.

Keywords: Generalized symmetries, differential forms, topological charges

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INTRODUCTION

Symmetries play a fundamental role in physics. One of the main contributions in this subject was given by Emmy Noether [1, 2], when she discovered a deep relation between symmetries and conserved quantities. The union of symmetries and quantum aspects has led to many unprecedented advances in the construction of physical theories describing from condensed matter systems to particle physics.

A new perspective on symmetries assigns a topological notion to the conserved charges. Usually charges are just conserved objects with no geometry at all. The charges can be reformulated in terms of topological closed surfaces, that can be linked or not with the charged objects. The topological nature of the surface allows us to change it smoothly, once we keep the defect inside. This interpretation is the starting point for the new forms of symmetries that we may find.

In the majority of cases, the symmetry operators act on local objects. However it is quite natural to construct extended objects in QFT, and to explore the underlying symmetries. One of the first works in that direction is due to Kalb and Ramond [3]. More recently, Kapustin and Thorngren also studied extended objects in topological field theories [4]. This triggered many subsequent interesting developments involving the study of symmetries in higher-order objects, like lines and surfaces. The breakthrough work of Gaiotto, Kapustin, Seiberg and Willett [5] layed out the basis of global generalized symmetries.

Generalized symmetries can be usually classified as i) higher-form; ii) subsystem; and iii) non-invertible symmetries [6]. A higher-form symmetry consists of acting with operators $U(\mathcal{M}^{D-p-1})$ on p-form (i.e., p-dimensional) extended objects, which leads to the conservation of a $(p+1)$-form current. Subsystem symmetries are intrinsically nonrelativistic. While in the higher-form the charge is topological and independent of the shape of the manifold $\mathcal{M}^{D-p-1}$, in the subsystem symmetry the charge depends on the coordinate of the chosen manifold, leading to a huge number of conserved charges[1]. This type of symmetry has a fundamental role in topological phases of matter known as fractons [7, 8]. The non-invertible symmetry, as the name says, has no notion of an inverse operation. This means that this type of symmetry is not implemented by an unitary operator. For example, the axial anomaly in QED can be reinterpreted in terms of a non-invertible symmetry [9]. These operators are mathematically described by a fusion category, not a group as in the previous symmetries. The main features of the symmetries are summarized in the Table 1.

[1] Proportional to the system size.

Properties of symmetry op.	**Ordinary symmetry**	**Higher-Form symmetry**	**Subsystem symmetry**	**Non-invertible symmetry**
Codimension in spacetime	1	> 1	> 1	≥ 1
Topological	yes	yes	not completely (conserved in time)	yes
Fusion rule	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	group $g_1 \times g_2 = g_3$	category $\mathcal{D} \times \mathcal{D}^\dagger \neq 1$

Tabela 1 – Properties of Generalized Global Symmetries [10].

In this work we are mainly concerned with symmetries of the higher-form type applied to QED. One of the consequences of considering this symmetry is that we can reinterpret the photon as a Goldstone boson of the spontaneous breaking of a higher-form symmetry. In $D = 3$ this result was originally obtained by Rosenstein [11, 12], where the symmetry that is broken is a $0-$form kind (ordinary).

This work is organized as follows. In Chapter 1 we make an introduction to the mechanism of spontaneous symmetry breaking. In Chapter 2 we construct the line objects that are charged under the higher-form symmetry, i.e., the Wilson and 't Hooft lines. In Chapter 3 we use the Wilson line VEV to find the possible phases of QED in various dimensions. Then we introduce the basic notions of higher-form symmetries in Chapter 4 and apply them to Maxwell theory in Chapter 5. Finally, in Chapter 6 we study the spontaneous breaking of the $1-$form electric symmetry, leading to the photon as the Goldstone boson.

The convention used in this work will be the natural units for $c = \hbar = 1$. We will also use the Einstein sum convention.

1 SPONTANEOUS SYMMETRY BREAKING

The mechanism of SSB is a very important notion used in many different areas of modern physics. Before we dive into the consequences and examples of this process let us make a quick overview of what leads a system to spontaneously break some symmetry.

1.1 SSB in Quantum Theory

Let us start with a symmetric Hamiltonian under the transformation $U = e^{i\alpha Q}$. Since it is a symmetry of the system, the states might be degenerate because of the relation

$$[Q, H] = 0.$$

It leads to different states connected by the symmetry with the same energy eigenvalue. It is valid for all states, so it is for the ground state. There is two different ways that the symmetry can act on them. In first place, we may have a singlet representation of the symmetry group, namely the state just change by a phase factor

$$Q\left|0\right\rangle = q\left|0\right\rangle \quad \Rightarrow \quad U\left|0\right\rangle = e^{i\alpha q}\left|0\right\rangle. \tag{1.1}$$

In the case where the generator Q anihilates the vacuum $Q\left|0\right\rangle = 0$ or has $q = 0$ we see that the new state is exactly the same. This is said to be a symmetric state, since it does not change under the symmetry.

The other realization is obtained when the states are not eigenstates of the symmetry operator,

$$U\left|0\right\rangle = \left|0'\right\rangle. \tag{1.2}$$

In this situation we still have degeneracy, that is $\langle [Q, H] \rangle = 0$, but now the states are no longer equivalent, they in fact change under the symmetry. It is said that they are non symmetric states. This phenomenon can be better viewed in the thermodynamic limit. We can take as an example the translational symmetric system of N noninteracting particles localized at each coordinate $\vec{x}$. The system is symmetric under translations of the particles, and the transformation acts on each one as

$$U_\alpha \left|\psi(\vec{x})\right\rangle = \left|\psi'(\vec{x} + \vec{\alpha})\right\rangle = \left|\psi'(\vec{x}')\right\rangle, \tag{1.3}$$

transforming the state into another one. We want to know what happens to such a system when it is taken to the thermodynamic limit. For this we take the complete state as being the product $|\psi\rangle = \otimes_{\vec{x}} |\psi(\vec{x})\rangle$ of each particle. With this we can calculate the amplitude of the system going from the initial state to the transformed one according to

$$\begin{aligned}\langle\psi'|\psi\rangle &= (\otimes_{\vec{x}'} \langle\psi'(\vec{x}')|)(\otimes_{\vec{x}} |\psi(\vec{x})\rangle) \\ &= \prod_{\vec{x}} \langle\psi'(\vec{x})|\psi(\vec{x})\rangle = \langle\psi'(\vec{x})|\psi(\vec{x})\rangle^N ,\end{aligned}$$

where we used the orthogonality of states in $\vec{x} \neq \vec{x}'$ [13]. If they are distinct, the product is $\langle\psi'(\vec{x})|\psi(\vec{x})\rangle < 1$. Taking the thermodynamic limit, $N \to \infty$, it goes to zero, making the states orthogonal. This is a very important conclusion, because if the ground states connected by the symmetry are orthogonal in the thermodynamic limit, so will be the excited states made up on them. It leads us to conclude that all of these excited states cannot be described by a single separable Hilbert space. It rather can be accommodated in n different equivalent Hilbert spaces, with n being the degeneracy [14]. The conclusion of this analysis is that the system with this behavior cannot be described by a linear realization of the symmetry as occurs in the case where $|0\rangle$ is an eigenstate of U, because it cannot be realized by a multiplet structure. Now different ground states lead to different Hilbert spaces in the thermodynamic limit, but at the same time the Hamiltonian is still symmetric.

Now we have various spaces to build our system, but how to know what ground state to choose? To answer this question we can introduce a new operator $\mathcal{O}$, which has the inequivalent ground states as eigenstates with different eigenvalues and its expectation value in a symmetric state is zero, that is

$$\langle 0|\mathcal{O}|0\rangle \begin{cases} = 0, & \text{if} \quad U|0\rangle = e^{iq}|0\rangle \\ \neq 0, & \text{if} \quad U|0\rangle = |0'\rangle \end{cases} .$$

This is exactly the behavior of an order parameter of phase transitions, in the symmetric phase its value is zero, and in the non symmetric phase the value is non vanishing. This transition is named spontaneous symmetry breaking. This is an emergent effect that is observable when the theory is taken to the thermodynamic limit, but it gives really interesting effects in the quantum realm.

Now let us investigate why the system goes spontaneously from one phase to another. This is only possible because in some moment the system has chosen one of the vacuum connected by the symmetry, why does it happen? Considering a kinetic Hamiltonian with a quadratic term in the momentum

$$H_{kin} \propto \int dx \vec{p}^{\,2}(x).$$

Making a Fourier transform in $\vec{p}(x)$ we have $\vec{p}(x) = \sum_k e^{i\vec{k}.\vec{x}} \vec{p}_k$, so the total momentum is given by

$$\vec{p}_{tot} = \int dx \vec{p}(x) = \int dx \sum_k e^{i\vec{k}.\vec{x}} \vec{p}_k = V \vec{p}_{\vec{k}=0},$$

with V being the volume of integration. Taking all this we have the Hamiltonian

$$\begin{aligned} H_{kin} &\propto \int dx \sum_{k,k'} e^{i\vec{x}.(\vec{k}+\vec{k}')}\vec{p}_k\vec{p}_{k'} = V\sum_k \vec{p}_{-k}\vec{p}_k \\ &= \frac{1}{V}\vec{p}_{tot}^{\,2} + V\sum_{k\neq 0}\vec{p}_{-k}\vec{p}_k. \end{aligned}$$

The first term corresponds to the total kinectic energy of the system as a whole, the second one is related to internal excitations.

This analysis gives us a hint of what happens to the whole system. The inequivalent symmetric states $|\psi\rangle$ given in the previous discussion are created up on the kinetic term of the Hamiltonian, because it is related with the whole configuration. These are called tower of states [13]. Due to the term of $1/V$ the spectrum of these states will be of the order $1/N$, since $V \propto N$. When it goes to the thermodynamic limit the energy eigenvalues get closer and closer, with the values converging all to the smaller one. This very little space between the states $|\psi\rangle$ make them really unstable, and any tiny perturbation can excitate one of them, making the system choose one of the inequivalent vacuum, spontaneously breaking the symmetry.

Given this introduction to the subject, we must enter in details in the next sections. Starting with a closer view on the order parameter. Then we will use as example a scalar theory with $U(1)$ symmetry. Later we will talk about Goldstone bosons, which are gapless excitations that arise in the theory after the SSB.

1.2 Order Parameter

The definition of symmetric states given by the action of the charge on it is not suitable in the thermodynamic limit, it is not well defined there, so we need to distinguish the phases with a different method. As we said, a proper way to distinguish the different ground states (the symmetric and non symmetric) is through the introduction of some order parameter $\mathcal{O}$.

Let us consider the symmetry operator $U = e^{i\alpha Q}$, with Q being the charge generator, and a generic local operator $\phi(x)$. The action of the symmetry on the operator is given by

$$\phi'(x) = U\phi(x)U^{\dagger}. \tag{1.4}$$

To know if we are in a symmetric phase or not it is intuitive to check whether the mean value of $\phi(x)$ keeps the same, that is $\langle\phi\rangle = \langle\phi'\rangle$. If it is satisfied then we are on the symmetric state. Otherwise we have a change in the operator indicating a non symmetric phase. So let us define the variation $\mathcal{O} = \delta\phi(x)$ where

$$\delta\phi(x) \equiv U\phi(x)U^{\dagger} - \phi(x). \tag{1.5}$$

Then we define

$$O(x) \equiv \langle \delta\phi(x) \rangle \, .$$

When $O(x) = 0$ there is no change in $\langle \phi \rangle$, implying that it is a symmetric phase. When $O(x) \neq 0$ then $\langle \phi' \rangle \neq \langle \phi \rangle$, and the symmetry is spontaneously broken. Expanding (1.5) up to first order in α, we find

$$O(x) = \langle [Q, \phi(x)] \rangle \, , \tag{1.6}$$

defining an order parameter [15]. If there exists any operator ϕ such that $\langle [Q, \phi(x)] \rangle \neq 0$, then the state is said to be in a spontaneously broken phase. If all operators lead to $\langle [Q, \phi(x)] \rangle = 0$ then the phase is symmetric. When we are dealing with quantum field theory, the operator $\phi(x)$ is the field charged under the symmetry and is called interpolating field.

1.2.1 Example: $U(1)$ symmetry

Now let us go to a concrete example of the symmetry breaking of the $U(1)$ group. Considering the complex scalar field $\phi(x)$ with Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - a\phi^*\phi - b(\phi^*\phi)^2, \tag{1.7}$$

with $b > 0$. We see that $\mathcal{L}$ is invariant under the phase transformation $\phi' = e^{i\alpha}\phi$, so this is a symmetry of the theory. Here we are interested in the vacuum state. We need to check if it is symmetric under the $U(1)$ transformation. In order to do this we need to see what is the order parameter for this symmetry [16].

For the $U(1)$ transformation we have the variation of the field $\delta\phi = i\alpha\phi$, so the order parameter can be defined as the mean value of the field

$$O(x) = \langle \phi(x) \rangle \, . \tag{1.8}$$

In a semiclassical regime we can find this expectation value (up to orders of $\hbar$) taking the minimum value of the classical energy ϕ_0

$$\langle \phi \rangle = \phi_0 + \mathcal{O}(\hbar). \tag{1.9}$$

So if we want to know the vacuum expectation value of ϕ we just need to find the classical ϕ_0 that minimizes the potential energy.

Giving the Lagrangian (1.7) we can identify the potential

$$V(\phi) = a\phi^*\phi + b(\phi^*\phi)^2, \tag{1.10}$$

which has the minimum at

$$\phi_0^*(a + 2b\phi_0^*\phi_0) = 0. \tag{1.11}$$

Considering that $b > 0$, we have two situations:

- $a > 0$: In this case the only way to satisfy (1.11) is to take $\phi_0 = 0$, so we have

$$\langle \phi \rangle = 0 \quad \longrightarrow \quad \text{Symmetric phase,}$$

- $a < 0$: In this case, eq. (1.11) admits a non trivial solution given by $\phi_0^* \phi_0 = -a/2b = v^2$, so the order parameter is

$$\langle \phi \rangle = v e^{i\theta} \quad \longrightarrow \quad \text{Spontaneously broken phase,}$$

where we used a phase factor for generality.

With it we see that the theory has a phase transition guided by the coefficient a.

Let us take a closer look into the broken phase. Here we have a configuration for the field that satisfies $\phi_0^* \phi_0 = v^2$, so if we put it into a complex parametrization $\phi_0 = \eta_1 + i\eta_2$, we find

$$|\phi_0|^2 = \eta_1^2 + \eta_2^2 = v^2. \tag{1.12}$$

That is, the ground state is composed of a curve with components η_1 and η_2, that can be viewed in the Figure 1, where the minimum of the potential is a curve. This is where the degeneracy is manifest. We have an infinity of combinations of η_1 and η_2 that satisfies the curve. Breaking the symmetry is equivalent to choose one of these vacuum and build the spectrum on it.

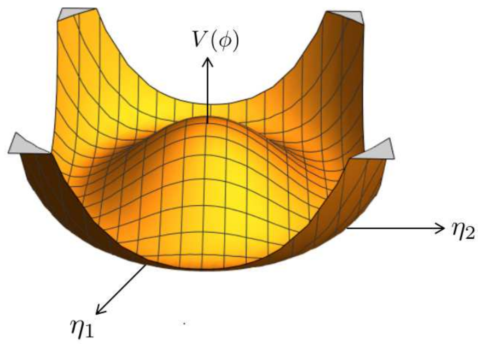

Figura 1 – Mexican hat potential in terms of fields η_1 and η_2.

Since we have freedom to choose any vacuum that satisfies (1.12), let us choose $\phi(x)$ in a parametrization that explicitly contains the two degrees of freedom, $\rho(x)$ and $\lambda(x)$, given by

$$\phi = \left(v + \frac{\rho(x)}{\sqrt{2}} \right) \exp\left(i \frac{\lambda(x)}{v\sqrt{2}} + i\theta \right), \tag{1.13}$$

with $\langle\rho(x)\rangle = \langle\lambda(x)\rangle = 0$. Substituting it on the Lagrangian we find

$$S = \int d^4x \left[\frac{1}{2}\left((\partial_\mu\rho)^2 + \left(1 + \frac{\rho}{v\sqrt{2}}\right)^2 (\partial_\mu\lambda)^2 - 2a\rho^2\right) - \sqrt{2}vb\rho^3 - \frac{b}{4}\rho^4\right]. \tag{1.14}$$

Now the theory is expressed with two new fields, the massive one $\rho(x)$ with mass $m^2 = 2a$ and the gapless $\lambda(x)$, which is called the Goldstone boson.

The action of the $U(1)$ symmetry on these fields can be obtained through eq. (1.13), giving

$$\rho' \to \rho$$
$$\lambda' \to \lambda + v\sqrt{2}\alpha.$$

The symmetry is realized non-linearly, i.e., the transformation is not implemented unitarily anymore. This is the non-linear realization of the symmetry. In other words the invariance is still in the broken phase, but it is realized in a different manner.

1.3 Nambu-Goldstone Modes

We just saw that the spontaneous break of a $U(1)$ symmetry gives us a massless boson. This is a consequence of the breaking of a continuous symmetry and can be stated in the following theorem [13]:

Theorem 1 *When a global continuous symmetry is spontaneously broken leaving a translation invariance intact, then there exists a mode in the spectrum whose energy vanishes when its wave number goes to zero.*

This is the Goldstone theorem, and these modes are called Nambu-Goldstone modes or Goldstone boson.

When we worked with the $U(1)$ symmetry in the semiclassical limit we found a massless particle rising in the theory. Let us see now how to find the quantum states of these excitations. First let us start with the order parameter, as we said before, when the symmetry is spontaneously broken it has a non vanishing value, and we will use it to find the state corresponding to the Goldstone boson. Consider a set of orthogonal states $|n, \vec{k}\rangle$ with energy $E_n(\vec{k})$ and momentum $\vec{k}$. We have the identity

$$\mathbb{I} = \sum_n \int \frac{d^Dk}{(2\pi)^D} |n, \vec{k}\rangle \langle n, \vec{k}| \,.$$

Using it in the order parameter we find

$$\langle 0|[Q, \phi]|0\rangle = \sum_n \int \frac{d^Dk}{(2\pi)^D} \Big(\langle 0| Q(t) |n, \vec{k}\rangle \langle n, \vec{k}| \phi |0\rangle - c.c\Big)\,. \tag{1.15}$$

Now we recall the definition of the charge in terms of the current

$$Q(t) = \int_\Omega d^D x j^0(x,t),$$

where the volume Ω is taken to be large but finite. Going back to (1.15) we get

$$\begin{aligned}\langle 0|[Q,\phi]|0\rangle &= \int_\Omega d^D x \sum_n \int \frac{d^D k}{(2\pi)^D} \left(\langle 0| j^0(x,t) |n,\vec{k}\rangle \langle n,\vec{k}| \phi |0\rangle - c.c\right) \\ &= \int_\Omega d^D x \sum_n \int \frac{d^D k}{(2\pi)^D} \left(\langle 0| e^{-i(Ht-\vec{P}.\vec{x})} j^0(0,0) e^{i(Ht-\vec{P}.\vec{x})} |n,\vec{k}\rangle \langle n,\vec{k}| \phi |0\rangle - c.c\right),\end{aligned}$$

where we have used the time evolution and space translation operators to rewrite the current. Considering that the vacuum has vanishing energy and momentum, we have

$$\begin{aligned}\langle 0|[Q,\phi]|0\rangle &= \int_\Omega d^D x \sum_n \int \frac{d^D k}{(2\pi)^D} \left(e^{i(E_n t-\vec{k}.\vec{x})} \langle 0| j^0(0,0) |n,\vec{k}\rangle \langle n,\vec{k}| \phi |0\rangle - c.c\right) \\ &= \sum_n \int d^D k \delta_\Omega(\vec{k}) \left(e^{iE_n t} \langle 0| j^0(0,0) |n,\vec{k}\rangle \langle n,\vec{k}| \phi |0\rangle - c.c\right) \neq 0.\end{aligned}$$

So in order to keep the behavior of the order parameter we need

$$\sum_n \int d^D k \delta_\Omega(\vec{k}) e^{iE_n t} \langle 0|j^0(0,0)|n,\vec{k}\rangle = \sum_n e^{iE_n t} \langle 0|j^0(0,0)|n,\vec{k}\to 0\rangle \neq 0. \tag{1.16}$$

That is, the action of the current $j^0(0,0)$ in the broken vacuum $|0\rangle$ creates a state that has a projection in an energy state $|n,\vec{k}\rangle$ when $\vec{k}\to 0$, so it creates a relevant excitation in the system.

The next thing to do is to check what energy state satisfies this relation. In order to do this let us use the condition of the independence of time of the order parameter, so we must have

$$\partial_0 \langle 0|[Q,\phi]|0\rangle = \sum_n \int d^D k \delta_\Omega(\vec{k}) iE_n \left(e^{iE_n t} \langle 0| j^0(0,0) |n,\vec{k}\rangle \langle n,\vec{k}| \phi |0\rangle - c.c\right) = 0.$$

The only way to satisfy this is $E_n \to 0$ when $\vec{k}\to 0$ for every n. In other words, the excitation created by the action of the current on the vacuum has a vanishing energy when the momentum goes to zero. This is the local excitation of the Goldstone boson, but we can consider the contribution of a superposition of plane waves creating the state

$$|\pi(\vec{k},t)\rangle \sim \int d^D x e^{i\vec{k}.\vec{x}} j^0(\vec{x},t) |0\rangle, \tag{1.17}$$

which has vanishing energy when the wave number goes to zero. In condensed matter this is said to be a gapless mode, and in particle physics it is said to be a massless particle (it has zero energy when it is not moving). The important point is that it is a consequence of the spontaneous breaking of a global continuum symmetry.

When we discussed the tower of states the important contribution was in $\vec{k}=0$, that is where the states are built. Here we have the contribution of the other modes,

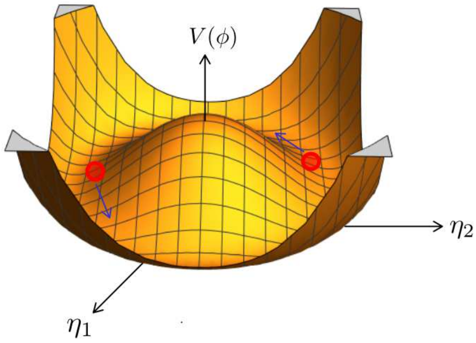

Figura 2 – Goldstone boson propagating in the vacuum of the Mexican hat.

the ones with $\vec{k} \neq 0$ or in the limit of $\vec{k} \to 0$. If we take the Mexican hat potential the Goldstone bosons can be viewed as excitations that propagate in the vacuum, connecting the different ground states.

Giving that the spontaneous break of the $U(1)$ symmetry gives birth to a massless mode in the theory, we can go back to the reparametrization (1.13) and identify the Goldstone boson as the massless field $\lambda(x)$.

1.3.1 Counting of Modes

That there are massless excitations in the SSB theories we already know, but how many? Let us discuss this now. When we derived the order parameter we used the vacuum that was changed by the operation of the charge, that is the non symmetric vacuum

$$|0'\rangle = Q\,|0\rangle\,. \tag{1.18}$$

There is something we kept hidden, that the charge can be composed of many different generators $Q = Q_a T_a$. So if we have N generators it is intuitive that we have N different ground states, but it is not that simple.

If we start with the symmetry group G we might not have all of the generators creating different ground states, some of them might keep the symmetry, so we say that the G is broken into the remaining symmetry group H. For example, we could have a $SU(2)$ symmetry being broken and leaving some $U(1)$ symmetry intact. Let us consider this example. Here we start with 3 generators of the group $SU(2)$ and find that only 1 keep the symmetry, that is

$$\begin{aligned} Q_1\,|0\rangle &= |0'\rangle \\ Q_2\,|0\rangle &= |0''\rangle \\ Q_3\,|0\rangle &= q\,|0\rangle\,. \end{aligned}$$

The number of broken generators is said to be $dim(SU(2)/U(1)) = 2$. Putting it back into the order parameter we may conclude that this is the number of Goldstone bosons that show up in the theory, but this is a very naive view of the system. In some cases it may happen that the states $|0'\rangle$ and $|0''\rangle$ are not linearly independent. In this case we would have only one boson rising from two different generators. This makes the counting of modes a little subtle. In the following, we give a simplified view of the construction to find the exact number of Goldstone excitations. The complete construction can be found in the references [13, 15, 17, 18].

In order to make the counting of modes, let us define the matrix

$$M_{ab} = -i \langle 0|[Q_a, j_b^0(x)]|0\rangle \,, \tag{1.19}$$

where $a, b = 1, 2, \cdots, dim(G/H)$, with G the symmetry group and H the remaining symmetry group. The matrix M measures the projection of the Goldstone boson state in another vacuum created by the charge Q_a. This matrix could have been built in the form $M_{ab} \sim \langle 0|Q_a j_b^0|0\rangle$, but it would account the contributions of $a = b$, and we are not interested on it, so we put the commutator to avoid it. If $M_{ab} = 0$, then the Goldstone boson created by one generator has no projection in the other vacuum, making the number of bosons equal to the number of broken generators. These are called type-A bosons. If $M_{ab} \neq 0$, then different generators give us the same boson. So we need to know how many degrees of freedom the matrix M has to be non vanishing, this will be the number of these bosons, which are called type-B.

Considering the matrix (1.19), then exists an orthogonal transformation O such that $\tilde{M} = OMO^T$ giving

$$\tilde{M} = \begin{pmatrix} M_1 & & & & & \\ & \ddots & & & & \\ & & M_p & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \quad M_i = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}. \tag{1.20}$$

In this form it is clearer to check what are the degrees of freedom of M. We can find it by seeing how many linearly independent vectors are needed to construct it. We see that we need $2p$ linearly independent vectors, which is the rank of the matrix,

$$\text{rank}\ \tilde{M} = 2p. \tag{1.21}$$

We already said that this matrix is a $dim(G/H) \times dim(G/H)$ object, and we said that when it is zero then we have a type-A boson. So by looking at $\tilde{M}$, we see that the number of null elements in the matrix will be the number of eigenvalues in the diagonal that are zero, that is

$$n_a = \dim(G/H) - \text{rank} M. \tag{1.22}$$

This is the number of Goldstone bosons of type-A that we have.

The number of type-B bosons will be the rank of M, but we have the anti-symmetric feature of the matrix, leaving us with only p degrees of freedom. So the number of type-B Goldstone bosons is

$$n_b = \frac{1}{2} rank M. \tag{1.23}$$

The main difference between type-A and type-B Goldstone bosons is their dispersion relations. Take the most general quadratic effective Lagrangian for the bosons π_a [13]

$$\mathcal{L}_{eff} = M_{ab}(\pi_a \partial_t \pi_b - \pi_b \partial_t \pi_a) + \bar{g}_{ab} \partial_t \pi_a \partial_t \pi_b - g_{ab} \nabla \pi_a \cdot \nabla \pi_b, \tag{1.24}$$

where $\bar{g}_{ab}$ and g_{ab} are coefficients. When $M_{ab} = 0$ we have two time and two space derivatives, that lead to the dispersion $\omega^2 \sim k^2$. That is, type-A bosons are relativistic. When $M_{ab} \neq 0$, the linear derivative in time will dominate, leading to the dispersion $\omega \sim k^2$, which is a non-relativistic behavior for the type-B bosons.

In conclusion, given the number of broken generators we can relate the relativistic and non-relativistic Goldstone bosons in the theory by

$$n_a + 2n_b = \dim(G/H). \tag{1.25}$$

2 LINE OPERATORS

Electromagnetism is one of the most important examples of a gauge theory. One thing about Quantum Electrodynamics (QED) is that it only shows up on nature (3+1 dimensions) in the Coulumb phase, i.e., it was never seen in some confining regime. In another dimensions this is not how it works, but how can we identify in what phase the theory lies? The main feature we can look at are the correlation functions of the theory, since they are capable to show us how their objects are related. The two main operators are Wilson and t' Hooft lines, and their expected value in the vacuum of the theory give us a hint about its phases.

Usually in Quantum Field Theory we deal with localized objects like fields $\psi(x)$, but nothing restrain us to work with non-local objects too. For example we could work with extended operators along a path line in space, which are known as a line operators. On the other hand we can deal with objects extended through time too, which are called line defects. There are two main objects like that in four dimensional QED, the Wilson and 't Hooft lines. The mathematics behind these lines is huge, and also is the physics. They play a very important role in defining phases of gauge theories and will be of great use in the course of this text. Let us briefly introduce them now and in the next chapters we will dive in the features of these objects, making more clear the need of them.

2.1 Wilson Line

Let us assume a field $\psi(x)$ that is build up on a spacetime manifold M. When it is invariant under some global transformation it has conserved charges that can be organized in an internal degree of freedom ω^i, where i runs through the number of generators. This is called the charge vector. The gauging of this theory is made when we take the transformation to be local, now each point $p \in M$ has a conserved charge vector $\omega^i(p)$ carried by the field $\psi(p)$. Thus for every point of $p \in M$ we have an internal space Ω_p where the charge vector $\omega^i(p)$ lives, like in the Figure 3. Making a gauge transformation is equivalent to rotate the charge vector in this space. The set of all spaces Ω_p together form another space, the fiber bundle [19, 20].

For example, the $U(1)$ gauge theory for electromagnetism creates 1-dimensional internal spaces for the gauge fields, so the charge vector only has one dimension and one charge. Whereas the $SU(2)$ gauge theory creates a 3-dimensional internal space due to the three generators, giving rise to vectors of charges $(\omega_1, \omega_2, \omega_3)$, which can be written in the three dimensional basis of generators.

The fiber bundle forms by itself another vector space, where we can move our

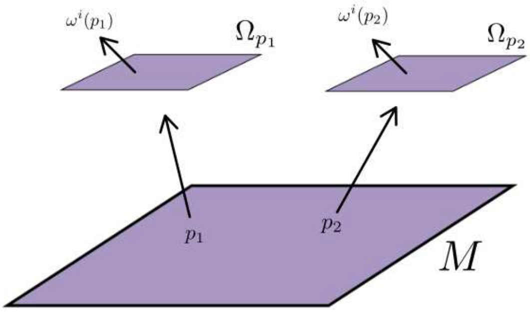

Figura 3 – For each $p \in M$ we have an associated internal space Ω_p where the matter fields live, and a gauge transformation is equivalent to a rotation in this space.

vectors [20]. The mechanism used to do this is the parallel transport. We relate different spaces with the covariant derivative $D_\mu = \partial_\mu - iqA_\mu$, that is, a charge vector ω^i is taken from one internal space to another by

$$\frac{dx^\mu}{d\tau} D_\mu \omega^i = 0 \quad \rightarrow \quad \frac{d\omega^i}{d\tau} = iq\frac{dx^\mu}{d\tau} A_\mu(x)\omega^i, \tag{2.1}$$

where $A_\mu(x)$ is the gauge field, q is the charge and τ the parameter of the curve which the path is taken [21]. The field $A_\mu(x)$ is the responsible for taking the initial charges $\omega^i(\tau_i)$ to the final $\omega^i(\tau_f)$, and by integrating[1] the equation of parallel transport we can relate then by

$$\omega^i(\tau_f) = W(x_f, x_i)\omega^i(\tau_i), \tag{2.2}$$

with

$$W(x_f, x_i) = \exp\left(iq\int_{x_i}^{x_f} A_\mu(x)dx^\mu\right), \tag{2.3}$$

the so called Wilson line.

We can extract the physical meaning of $W(x_f, x_i)$ taking its vacuum expectation value in the vacuum

$$\begin{aligned}\langle W(x_f, x_i)\rangle &= \langle \exp\left(iq\int_{x_i}^{x_f} A_\mu(x)dx^\mu\right)\rangle \\ &= \langle \exp\left(iq\int d^3y \int_{x_i(t_i)}^{x_f(t_f)} dt\frac{dy^\mu}{dt}A_\mu(y)\delta^{(3)}(\vec{x}-\vec{y})\right)\rangle = \langle \exp\left(i\int d^4x A_\mu(x)J^\mu\right)\rangle,\end{aligned} \tag{2.4}$$

so we conclude that the Wilson line through the path between x_i and x_f represents the trajectory of a probe particle between x_i and x_f, parametrized by a conserved current J^μ. This interpretation will be used in the next section to compute correlations in closed paths.

[1] This is only possible to do in the Abelian case, when we are in the non-Abelian case, we need to use path ordering because the generators do not commute, so it is not so simple to integrate them.

Now let us see how this object transforms under gauge transformations [22]. Taking the general transformation for $A^\mu(x)$ as

$$A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x), \tag{2.5}$$

and considering an infinitesimal path for the line $W(x+\epsilon, x) = \exp(iq\epsilon^\mu A_\mu(x))$, we have

$$\begin{aligned} W'(x+\epsilon, x) &= 1 + iq\epsilon^\mu \left(U(x)A_\mu(x)U^\dagger(x) + \frac{i}{q}U(x)\partial_\mu U^\dagger(x) \right) \\ &= [(1+\epsilon^\mu\partial_\mu)U(x)]U^\dagger(x) + iq\epsilon^\mu U(x)A_\mu(x)U^\dagger(x) \\ &= U(x+\epsilon)U^\dagger(x) + iq\epsilon^\mu U(x)A_\mu(x)U^\dagger(x) \\ &= U(x+\epsilon)(1 + iq\epsilon^\mu A_\mu)U^\dagger(x). \end{aligned}$$

Up to $\mathcal{O}(\epsilon^2)$ terms, we can write $W'(x+\epsilon, x) = U(x+\epsilon)W(x+\epsilon, x)U^\dagger(x)$, so for a path between x and y we have

$$W'(y, x) = U(y)W(y, x)U^\dagger(x). \tag{2.6}$$

When $x = y$ we have a similarity equation, which corresponds to a charged particle traveling a closed path in the space. When talking about the fiber bundle we can classify the Wilson loop as a holonomy, since it is a phase obtained by a whole loop around the fibers. Since the definition of holonomy requires that the field returns to the same point, we know that this object must be gauge invariant, since a frame change will not change the physical meaning. Taking all of this in consideration we see that the invariant object is

$$W(C) = \exp\left(iq \oint_C A_\mu dx^\mu \right). \tag{2.7}$$

In the non-Abelian case, this expression may be changed by a path ordering, taking the form $W(C) = Tr\left[igP\exp\left(\oint_C A_\mu dx^\mu\right)\right]$.

2.2 't Hooft Line

Now that we have introduced the Wilson line, which is equivalent to an electric charge traveling a path, consider the magnetic analog, which leads us to the so-called 't Hooft line. But before the discussion about the line, we need to understand how a magnetic charge can be obtained and what is its mathematical meaning.

2.2.1 Dirac Quantization

From the Maxwell equations of electromagnetism we conclude that there is no magnetic charges, since $\nabla.\vec{B} = \nabla.(\nabla \times \vec{A}) = 0$. But we could put some magnetic charge q_m by hand introducing a new magnetic flux through a surface S

$$\int_S d\vec{S}.\vec{B} = q_m. \tag{2.8}$$

If the vector potential $\vec{A}$ has some singularity in the volume enclosed by the surface S, then the gradient of the rotational will not vanish, with the point of singularity being where the monopole lies.

Since the monopole has a specified position and charge we can take an electric charged particle through a closed path around the singularity, which changes the free field by a phase

$$\psi \to e^{iq_e \oint \vec{A}.d\vec{x}}\psi, \tag{2.9}$$

that is known as the Aharonov-Bohm phase. Using (2.8) and the Stokes theorem we can integrate in the S region in Figure 4-(a), finding

$$\oint \vec{A}.d\vec{x} = \int d\vec{S}.\vec{B} = \frac{\Omega q_m}{4\pi}, \tag{2.10}$$

with Ω being the solid angle of the surface. Then the phase will be $\psi' = e^{i\frac{\Omega q_m q_e}{4\pi}}\psi$. The freedom to choose the path of integration allows us to make the integration by the other side of the sphere, like represented in Figure 4-(b), leading us to the phase

$$\alpha = -\frac{(4\pi - \Omega)q_m}{4\pi}.$$

Figura 4 – Spherical region enclosing the magnetic monopole.

Since both ways must lead to the same result, we expect that they differ by an integer

$$\frac{q_e q_m \Omega}{4\pi} + \frac{q_e q_m (4\pi - \Omega)}{4\pi} = 2\pi n.$$

Recalling that $q_e \in \mathbb{Z}$, we get the Dirac quantization of magnetic charges

$$q_e q_m = 2\pi n, \quad n \in \mathbb{Z}. \tag{2.11}$$

We can make this calculation a little more explicit taking the form of the fields in two different regions [23]. Since we are considering a new magnetic charge in the system, the Gauss law for the magnetism gives

$$\nabla.\vec{B} = q_m \delta^3(x), \tag{2.12}$$

which, in spherical coordinates and up to a gauge transformation, implies in a vector field

$$\vec{A} = \frac{q_m}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\hat{\varphi}. \tag{2.13}$$

It is clear that there is a singularity in $\theta = 0$, so all of the points in the sphere that have $\theta = 0$ are points of singularity. To solve this problem we may use two new fields, that are defined in different regions, $\vec{A}_+$ on the top of the sphere, and the $\vec{A}_-$ on the bottom, considering that in the equator of the sphere they change only by a gauge transformation. They are

$$\vec{A}_+ = \frac{q_m}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\hat{\varphi}, \quad 0<\theta<\pi/2, \tag{2.14}$$

$$\vec{A}_- = \frac{q_m}{4\pi}\frac{-1-\cos\theta}{r\sin\theta}\hat{\varphi}, \quad \pi/2<\theta<\pi. \tag{2.15}$$

Together, they lead to the correct $\vec{B}$ field everywhere. Since these two fields need to produce the same result, they must be connected by a gauge transformation in the overlapping region

$$\begin{aligned} \vec{A}_- - \vec{A}_+ &= \nabla f \\ -\frac{q_m}{2\pi r\sin\theta} &= \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi} \quad \Rightarrow \quad f = -\frac{q_m}{2\pi}\varphi. \end{aligned}$$

Therefore we identify the gauge transformation as

$$\psi'(x) = e^{-iq_e q_m\varphi/2\pi}\psi(x). \tag{2.16}$$

Since the group $U(1)$ in compact, when it gives a full turn the wave function must return to the same as it was before, namely,

$$\begin{aligned} \varphi = 0: &\quad \psi'(x) = \psi(x) \\ \varphi = 2\pi: &\quad \psi'(x) = \psi(x). \end{aligned}$$

To satisfy this we must have $q_e q_m = 2\pi n$, with $n \in \mathbb{Z}$, the Dirac quantization condition.

2.2.2 Dual Vector Field

Now that we have the description of a magnetic charge, we can go forward on the construction of the line that describes its behavior through a path, just like the electric case with the Wilson line. It is well known that the Maxwell equations in the absence of charges are invariant under the duality transformation given by

$$\begin{aligned} \vec{E} &\to -\vec{B} \\ \vec{B} &\to \vec{E}. \end{aligned}$$

In differential forms these fields can be written using the Faraday tensor given by a 2-form[2]

$$F = dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu, \tag{2.17}$$

[2] The properties of differential forms are given in Section 4.1.

with $F^{0i} = -E^i$ and $F^{ij} = \epsilon^{ijk}B_k$. Taking the duality transformation is equivalent to use a new Faraday tensor that satisfies $F_D^{0i} = B^i$ and $F_D^{ij} = \epsilon^{ijk}E_k$. It can be obtained by taking the Hodge dual of the 2-form F [24]

$$F_D = *F = *dA. \tag{2.18}$$

If for the electric charge we constructed a line with the vector field that generates it, for the magnetic one the process will be essentially the same, but now the vector field will be the dual one, defined as

$$F_D \equiv d\bar{A}. \tag{2.19}$$

So in differential forms we have the Wilson $W(C)$ and 't Hooft (magnetic) $T(C)$ lines as

$$W(C) = \exp\left[iq_e \int_C A\right] \quad \text{and} \quad T(C) = \exp\left[iq_m \int_C \bar{A}\right]. \tag{2.20}$$

This construction was motivated by the electromagnetic case, but it can be made for any Abelian gauge theory. The vector gauge field is defined as a 1-form object and its external derivative gives rise to the Faraday tensor F, a 2-form. In turn, the dual of F can be defined using a dual form $F_D = d\bar{A}$, and the theory also has Wilson and 't Hooft lines. This is just the first approach made in $D = 3 + 1$. In next chapters we will see that in other dimensions the 't Hooft operator is not always a line, it depends on the dimension where the system is embedded.

Some may ask how the two objects are related in electromagnetism. To understand this we can compute the 't Hooft and Wilson loops and see what happens when their paths are crossed. This will give us a commutation relation. In purpose to relate the two objects we will put the 't Hooft loop in terms of canonical variables of the field A using the relation

$$*dA = d\bar{A} \quad \Rightarrow \quad \frac{1}{2}\epsilon_{\mu\nu}{}^{\sigma\rho}\partial_\sigma A_\rho dx^\mu \wedge dx^\nu = \partial_\mu \bar{A}_\nu dx^\mu \wedge dx^\nu. \tag{2.21}$$

Taking the conditions of canonical quantization of A^μ, $A^0 = 0$ and fixed time, and the Stokes theorem[3], the line integral of $\bar{A}$ gives:

$$\begin{aligned}\oint_{C=\partial S} \bar{A} &= \int_S d\bar{A} \\ &= \frac{1}{2}\int_S \epsilon_{jl}{}^{0i}\partial_0 A_i dx^j \wedge dx^l \\ &= -\frac{1}{2}\int_S \epsilon^i{}_{jl} E_i dx^j \wedge dx^l.\end{aligned}$$

Thus in vector notation we get

$$T(C) = \exp\left[-iq_m \int_S dS^i E_i\right]. \tag{2.22}$$

[3] The Stokes theorem in differential forms is given by $\oint_{\partial S} A = \int_S dA$

Now that we have the lines in terms of canonical variables, the commutation relation can be calculated using the BCH theorem $e^a e^b = e^{[a,b]} e^b e^a$, leading to

$$W(C)T(C') = e^{-q_m q_e [\oint_C A, \oint_{C'} \bar{A}]} T(C')W(C). \tag{2.23}$$

Since the variables $A^i(x)$ and $E^j(y)$ are canonical conjugate we have the relation

$$[A_i(\vec{x}), E_j(\vec{y})] = -i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}). \tag{2.24}$$

Using (2.20), (2.22) and (2.24) it follows that

$$\begin{aligned} -q_e q_m \left[\oint_C A, \oint_{C'} \bar{A}\right] &= -q_e q_m \left[\oint_C dx^i A_i(x), \int_{S'} dS_i(y) E^i(y)\right] \\ &= i q_e q_m \int_{S'} dS_i(y) \oint_C dx^i \delta^{(3)}(\vec{x}-\vec{y}). \end{aligned} \tag{2.25}$$

This can be interpreted as a linking number. When the curve C pierces the surface S' this is non vanishing, but when it does not it is zero, as in the Figure 5.

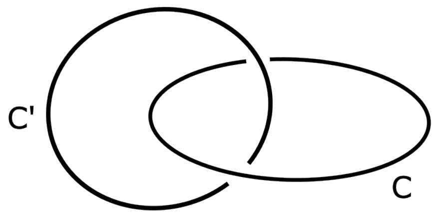

Figura 5 – Intersection between the paths C' and C.

Considering that the two curves are linked, we can put it back to the commutation relation

$$W(C)T(C') = e^{i q_e q_m \int_{S'} dS_i(y) \oint_C dx^i \delta^{(3)}(\vec{x}-\vec{y})} T(C')W(C)$$

and add the linking to the 't Hooft line, redefining it as

$$T(C') = \exp\left[-i q_m \int_{S'} dS'_i(y) \left(E^i(y) - q_e \oint_C dx^i \delta^{(3)}(\vec{x}-\vec{y})\right)\right]. \tag{2.26}$$

This is exactly the addition of an electric flux produced by a particle traveling the curve C. One line induces a new flux to the area enclosed by the other, in this case the Wilson line induced an electric flux measured by the 't Hooft line. Since this linking is only proportional to integer number of pierces, and the Dirac quantization condition (2.11) stands for the charges, we conclude that

$$W(C)T(C') = T(C')W(C). \tag{2.27}$$

Before closing the section, we note that for every dimension the gauge field is described by a 1−form object A and the field strength is defined as $F = dA$, being F a

2−form. When we consider the dual field we keep with the previous definition given by (2.19), which F_D is a $(D-2)-$form, implying that the field $\bar{A}$ is a $(D-3)-$form. In the previous case we were using $D = 4$, so the 't Hooft operator was a line, given by a 1−form object. Now if we go to $D = 3$ the dual field is a scalar, giving rise to a local scalar 't Hooft operator.

2.3 Area and Perimeter Law

Now that we have both Wilson and 't Hooft lines, we can discuss about phases of gauge theories. To better describe the phases of a physical system we need to understand how the interaction is manifest in that phase. In the case of gauge theories we need to understand how charged objects interact through the potential in the given phase. The better way to do that is taking the vacuum expectation value (VEV) of Wilson (or 't Hooft) lines.

Take as an example the case of two local operators $\mathcal{O}(x)$ and $\mathcal{O}(0)$ [25, 26]. Taking the correlation of them in the vacuum may lead essentially to two distinct results. First the correlated one

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \propto e^{-|x|/\alpha}, \tag{2.28}$$

with $[\alpha] = -1$ in mass dimension. It shows us that the average of the two objects localized on the edges of a line has a short-range correlation, i.e., it decays very fast with the distance through the exponential. This is the case of a confined system, because de probability to find the objects out of the region $|x| < \alpha$ is vanishing, so the objects are confined. We also may have another configuration

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \propto \langle \mathcal{O}(x)\rangle \langle \mathcal{O}(0)\rangle , \tag{2.29}$$

the non-correlated one. In this case we see that the correlation between the two objects depends only on the external points of the path in between. This is the case where we say the system is free, or non-confining, because the correlation is independent of the path. If we consider this example in a higher dimensional case, for example using a line operator, the confining case is proportional to the "area"of the line and the free case is proportional to the "perimeter".

Now let us think about QED. What describes its behavior is the correlation between charged objects, so in order to make the same approach as in the example we need to calculate the correlation between an electron and a positron. Going back to expression (2.4) we concluded that the VEV of a Wilson line is equivalent to some current J^μ propagating through the curve. Let us determine this curve as in the Figure 6, since the object is extended through spacetime, we can create a closed path Γ with time T and distance R. In this configuration we have a particle propagating in positive time direction and one propagating on negative time, so the system is equivalent to one electron and one positron being created at $t = 0$ separated by the distance R, propagating in a time interval T and then being annihilated. Given this interpretation we can calculate $\langle W(\Gamma)\rangle$ to see what are the phases of QED.

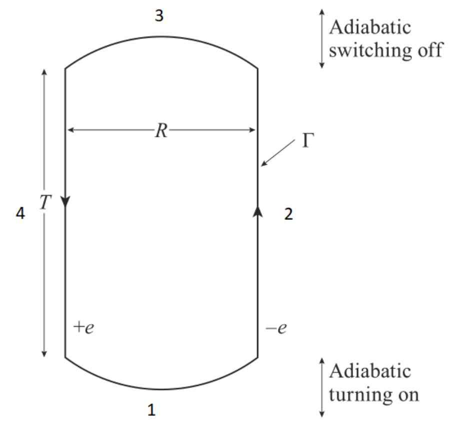

Figura 6 – Path Γ for the Wilson loop, with R the spatial distance and T the time interval. [26]

Before we make the explicit calculation of $\langle W(\Gamma)\rangle$ let us examine what results we might obtain from it. In order to do this we will use the fact that, in the limit of $T \to \infty$, we have

$$\langle W(\Gamma)\rangle \propto e^{-V_{int}T}. \tag{2.30}$$

In conclusion, the calculation of the $\langle W(\Gamma)\rangle$ gives us the interaction potential between charges. The expression above is better explained in the Appendix D.

It is helpful to compare with the previous example of correlations between local operators. There we had a local object and we tested its correlation through points localized at the edges of a curve. Here we have an extended operator and we are calculating it through a closed curve, extending it to a two dimensional case we have area and perimeter laws given by [21]

$$\begin{aligned} \text{Area law:} \quad & \langle W(\Gamma)\rangle \propto e^{-A(\Gamma)} \to \text{Confining phase} \\ \text{Perimeter law:} \quad & \langle W(\Gamma)\rangle \propto e^{-P(\Gamma)} \to \text{Higgs phase,} \end{aligned}$$

where $A(\Gamma)$ and $P(\Gamma)$ denotes the area and perimeter given by the curve Γ. Besides these two types of phases there is another one, the so-called Coulomb phase, which is scale-invariant dependent [27, 21].

Confining Phase: Using (2.30) we see that to keep the area dimension of the confining regime we have $A = RT$, so the potential will be

$$V_{conf} \propto \sigma R, \tag{2.31}$$

with σ being the string tension. This potential leads to a growing interaction between the particles as they are separated, increasing the energy of them. It is the case of QCD, where we have a confining regime.

Higgs Phase: For the perimeter law on the (2.30) we have $P = (T + R)/2$, so we need a potential of the form

$$V_{Higgs} \propto \xi, \tag{2.32}$$

where ξ is a constant [26]. The energy to separate two charges is finite here. This is a deconfined phase. In this regime the gauge bosons are massive, which limits their range, making the interaction weak. This is known as the Higgs phase.

Coulomb Phase: We saw what happens for correlations proportional to area and perimeter, but what about correlations that are scale-invariant? This is a situation where the correlation is constant, so it is not free neither is weakly coupled, but the intensity is small enough to do not let it confine. Another feature of this case is that it is scale invariant, so in this phase the theory is the same for large and small scales. In order to keep it constant on the (2.30) we may have

$$\langle W(\Gamma)\rangle \propto e^{-\alpha\left(\frac{T}{R}+\frac{R}{T}\right)},$$

so, in the limit of $T \to \infty$, the potential has to be

$$V_{Coulomb} \propto \frac{\alpha}{R}, \tag{2.33}$$

with α being a constant with no mass dimension.

2.4 Computation of VEV of Wilson loop

Now that we have the description of the phases we might find, let us go to the explicit calculation of $\langle W(\Gamma)\rangle$ in D dimensions [22, 26] and see what phases are present in QED. Using Euclidean space-time, we can explicitly calculate

$$\langle W(\Gamma)\rangle = \int \mathcal{D}A \exp\left\{\int d^D x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right)\right\}, \tag{2.34}$$

where we used the free action for the potential field $S = \int d^D x \frac{1}{4} F^{\mu\nu}F_{\mu\nu}$. Using the result obtained in the Appendix A, we get

$$\langle W(\Gamma)\rangle = \exp\left[\frac{q_e^2}{2}\oint_\Gamma dx_\mu \oint_\Gamma dy_\nu \Delta^{\mu\nu}(x-y)\right], \tag{2.35}$$

with $\Delta^{\mu\nu}(x-y)$ being the Euclidian propagator of the massless photon given by

$$\Delta^{\mu\nu}(x-y) = g^{\mu\nu}\int \frac{d^D p}{(2\pi)^D}\frac{e^{ip_\mu(x^\mu - y^\mu)}}{p^2},$$

where $\mu = 1, ..., D$. The momentum integral can be explicitly computed and gives

$$\Delta^{\mu\nu}(x-y) = g^{\mu\nu}\frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{D/2}|x-y|^{D-2}}. \tag{2.36}$$

In order to find the Wilson loop in D dimensions we need to solve the integral

$$\langle W(\Gamma)\rangle = \exp\left[\frac{q_e^2}{2}\frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{D/2}}\oint_\Gamma dx_\mu \oint_\Gamma dy^\mu \frac{1}{|x-y|^{D-2}}\right]. \tag{2.37}$$

The first thing to do is to separate the path Γ in the four steps given in Figure 6,

$$\oint_\Gamma dx_\mu \oint_\Gamma dy^\mu \frac{1}{|x-y|^{D-2}} = \left[\int_1 dx_\mu + \int_2 dx_\mu + \int_3 dx_\mu + \int_4 dx_\mu\right] \times$$
$$\times \left[\int_1 dy_\mu + \int_2 dy_\mu + \int_3 dy_\mu + \int_4 dy_\mu\right]\frac{1}{|x-y|^{D-2}}.$$

Since we are considering the limit of $T >> R$ the integrals over paths 1 and 3 can be neglected [26]. So the only integrals left are the 2 and 4 made on the time direction x_D and y_D

$$\oint_\Gamma dx_\mu \oint_\Gamma dy^\mu \frac{1}{|x-y|^{D-2}} = \left[\int_2 dx_D + \int_4 dx_D\right]\left[\int_2 dy_D + \int_4 dy_D\right]\frac{1}{|x-y|^{D-2}}.$$

Now we have to analyze what contribution corresponds to the interaction between electron and positron. When the integrals are taken on the same path it means that we are calculating self-interaction, and it does not contribute to what we are seeking. In this case, using the symmetry between the variables, we can use only the terms

$$\oint_\Gamma dx_\mu \oint_\Gamma dy^\mu \frac{1}{|x-y|^{D-2}} = 2\int_{-T/2}^{T/2} dy_D \int_{-T/2}^{T/2} dx_D \frac{1}{|x-y|^{D-2}}.$$

Using $|x-y|^2 = (x_D - y_D)^2 + R^2 = R^2(t^2+1)$, with $(x_D - y_D)/R = t$, we get

$$2\int_{-T/2}^{T/2} ds \int_{(-T/2+s)/R}^{(T/2-s)/R} \frac{dt}{(t^2+1)^{(D-2)/2}}\frac{1}{R^{D-3}} = 2\int_{-T/2}^{T/2} ds \int_{-\infty}^{\infty} \frac{dt}{(t^2+1)^{(D-2)/2}}\frac{1}{R^{D-3}},$$

where we took the limit of $T/R \to \infty$. Putting this integral em terms of Gamma functions and going back to (2.37) we find

$$\langle W(C)\rangle = \exp\left[\frac{q_e^2}{2}\frac{\Gamma\left(\frac{D-2}{2}\right)}{4\pi^{D/2}}\frac{2T\sqrt{\pi}}{R^{D-3}}\frac{\Gamma\left(\frac{D-3}{2}\right)}{\Gamma\left(\frac{D-2}{2}\right)}\right] = \exp\left[\frac{2q_e^2}{4\pi^{(D-1)/2}}\frac{\Gamma\left(\frac{D-1}{2}\right)}{(D-3)}\frac{T}{R^{D-3}}\right],$$

which leads to a potential energy of the form

$$V_{int} = -\frac{\Gamma\left(\frac{D-1}{2}\right)}{2\pi^{(D-1)/2}(D-3)}\frac{q_e^2}{R^{D-3}}. \tag{2.38}$$

In the next chapter we will use this potential in different dimensions and see what phases the electromagnetism manifest in each dimension. Something that can be easily inferred is that for $D = 3$ this expression diverges, so we will fix it in the next chapter.

3 PHASES OF QED

We have discussed the area and perimeter laws for the Wilson loop in the previous chapter. Now we are going to implement this on QED in several dimensions and see what are the phases of the theory in each dimension.

3.1 $D = 2$

In the two dimensional case the field strength has only one component F^{01}, which leads to the action

$$S = \int d^2x \left(-\frac{1}{2e^2} F_{01} F^{01} + A_\mu J^\mu \right), \tag{3.1}$$

where we are using the factor of e^2 to keep the right dimension [21]. In D dimensions we still have the vector field as a $1-$form, so it has $[A^\mu] = 1$. Considering the derivative term in the field strength, the term $F^{\mu\nu} F_{\mu\nu}$ has dimension 4. In order to keep the action dimensionless we add the term $1/e^2$ with $[e^2] = 4 - D$ to the kinetic part. In $D = 4$ it is often suppressed because it has no dimension, but in lower dimensions it is important, so we will keep it here with $[e^2] = 2$. To keep it consistent with the previous discussion in (2.38) let us use $q_e = ne$, with $n \in \mathbb{Z}$.

Usually when we are solving the wave equations for the vector potential we use the gauge fixing condition and the equation of motion to fix two degrees of freedom. This is what leads to two transverse polarization modes in $D = 4$, namely, the two physical degrees of freedom. In D dimensions the analysis is essentially the same, and we have $D-2$ physical degrees of freedom. In $D = 2$ there is no electromagnetic wave propagation, because it has no freedom to propagate. It can be obtained in terms of the equation of motion for the free case

$$\partial_1 F^{01} = \partial_0 F^{01} = 0,$$

which leads to a constant electric field, with no propagation modes.

When we add matter the behavior of the field changes. Considering that we put a point charge in the origin with discrete charge n, we can calculate the new electric field by

$$\frac{1}{e^2} \partial_1 F^{01} = n\delta(x) \quad \Rightarrow \quad F^{01} = ne^2 \theta(x) + \varepsilon, \tag{3.2}$$

where ε is a constant electric field and $\theta(x)$ is the step function, with $\theta(x > 0) = 1$ and $\theta(x < 0) = 0$. Now let us put a charge n in $x = -L/2$ and a charge $-n$ in $x = L/2$, and calculate the potential energy between these charges. To do that we will use that the energy stored in the field is

$$V_{int} = \int dx \frac{1}{2e^2} F_{01}^2.$$

Using

$$\frac{1}{e^2}\partial_1 F^{01} = n[\delta(-L/2) - \delta(L/2)] \quad \Rightarrow \quad F^{01} = \begin{cases} ne^2, & x \in (-L/2, L/2) \\ 0, & \text{otherwise} \end{cases}$$

we can calculate the integral over x in the interval $-\infty < x < \infty$. Doing this we find the energy

$$V_{int} = \frac{n^2 e^2}{2} L, \tag{3.3}$$

exactly what we found in (2.38) for $D = 2$. This behavior for the energy ensures confinement for the charges. The electric field forms a flux tube in the spatial dimension and confines the particles, because it has no other dimension to spread.

3.1.1 Vortices

Now some may ask what happens if we couple the electromagnetic field to a Higgs potential. What is the broken phase? The answer is that the system keeps its confining regime. The action of the coupling with a complex scalar field is [21]

$$S = \int d^2x \left(\frac{1}{2e^2} F_{01}^2 + |\mathcal{D}_\mu \phi|^2 - m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4 \right), \tag{3.4}$$

with $\mathcal{D}_\mu \phi = \partial_\mu \phi - iA_\mu \phi$, giving that the charge of the scalar field is $n = 1$ and $\lambda > 0$. Here we are interested in the limit of $|m^2| \gg e^2$, that can be read at two different regimes, $m^2 \gg e^2$ or $m^2 \ll -e^2$. Next we will see how the system is manifest in these two different conditions.

The action given above is $U(1)$ gauge invariant, but if we take the vacuum of the theory it might not be. Taking the minimization condition for the potential we find

$$(m^2 + \lambda|\phi_0|^2)\phi_0 = 0.$$

There is only two ways for it to be satisfied, if $m^2 > 0$ then $\phi_0 = 0$ and the only positive solution is when $m^2 \gg e^2$. This is the symmetric phase, the vacuum is unique and the theory behaves in the same way as discussed before. The scalar field generates particles and anti-particles of mass m that interact through a confining potential, just as in (3.3).

The second possible solution is when $m^2 < 0$. In this case we find $|\phi_0|^2 = -m^2/\lambda$ and the valid regime is for $m^2 \ll -e^2$. This is not a particle scenario anymore, since now the symmetry is broken and the action can be rewritten in Euclidean space as

$$S = \int d^2x \left(\frac{1}{2e^2} F_{12}^2 + |\mathcal{D}_i \phi|^2 + \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \right), \tag{3.5}$$

where $i = 1, 2$ and $v^2 = -m^2/\lambda$. For this action to be finite we need to put some conditions on the limit $r \to \infty$, namely,

$$\mathcal{D}_i\phi = 0, \quad \text{and} \quad |\phi| = v, \quad r \to \infty. \tag{3.6}$$

This implies that we will have an asymptotic solution of the form $\phi = ve^{i\theta}$. By imposing this limit to the action we restrict our solution to a map

$$e^{i\theta} : S^1 \to S^1$$

that goes from a S^1 spatial manifold to a S^1 manifold where the field lives. As we can see in Appendix B this map forms an equivalence class between solutions that winds around the manifold. What differs one solution to another is the winding number. To find the expression let us use the parametrization $e^{i\theta} = (x, y)$ with $x = \cos\theta$ and $y = \sin\theta$, which using (B.4) leads to [16]

$$n = \frac{1}{2\pi}\oint(xdy - ydx) = \frac{1}{2\pi}\oint d\theta. \tag{3.7}$$

Now taking the variation of ϕ, given by $d\phi = i\phi d\theta$ and substituting in the winding number we find

$$n = \frac{1}{2\pi i}\oint\frac{\phi^* d\phi}{\phi^*\phi}, \tag{3.8}$$

which is called vorticity. To differ the inequivalent solutions let us rewrite our fields in terms of the vorticity n

$$\phi = ve^{in\theta}, \tag{3.9}$$

where $n \in \mathbb{Z}$, because for different directions of contour we have different signs of vorticity [21]. The solutions $n > 0$ are called vortices and $n < 0$ anti-vortices.

We may also want to relate this solution with the gauge field. Using the conditions (3.6) we have in differential form

$$d\phi = iA\phi \quad \to \quad \frac{\phi^* d\phi}{\phi^*\phi} = iA,$$

so we find

$$n = \frac{1}{2\pi}\oint A = \frac{1}{2\pi}\int_S F = \Phi. \tag{3.10}$$

We get that the vorticity is proportional to the magnetic flux going through the surface enclosed by the curve. It can be clearer if we put the system in a three dimensional space-time manifold, and consider this solution as a static vortex. The vortices are responsible for keeping the confining regime in bidimensional QED.

Following the previous discussion we will calculate the Wilson loop expectation value in a squared path to check what is the potential in the presence of vortices [28]. Now instead of making the path integral over the configurations of the field A we will sum over the vortex μ, so for one vortex configuration we have the line $\left(e^{iq\oint A}\right)_{vort} = \int d\mu_{vort} e^{iq\oint A}$ [28]. We also need to consider the anti-vortex configurations, so we have the complete contribution for one line given by

$$\left(e^{iq\oint A}\right)_{vort+anti-vort} = \int (d\mu_{anti-vortd} + d\mu_{vort}) e^{iq\oint A} = \int d\mu e^{iq\oint A}, \tag{3.11}$$

with

$$d\mu_{vort/anti-vort} = \mu^2 d^2 x_0 e^{-S_{vort/anti-vort}}, \tag{3.12}$$

where $e^{-S_{vor}}$ is the first order contribution to the action, μ^2 is a factor coming from the vortex measure and its value is irrelevant for our purposes, and x_0 are the coordinates of the vortex. Now summing over the infinite configurations of identical lines $\left(e^{iq\oint A}\right)_{vort+anti-vort}$ we find the complete Wilson line

$$\langle e^{iq\oint A}\rangle = \frac{\sum_p \frac{1}{p!}\left(e^{iq\oint A}\right)^p_{vort+anti-vort}}{\sum_p \frac{1}{p!}\left(\int d\mu\right)^p}. \tag{3.13}$$

In order to find the final form of the line let us calculate the one vortex contribution first. Considering that $\oint A = 2\pi n$, and making the integration in d^2x_0 over the square given by space L and time T we have

$$\left(e^{iq\oint A}\right)_{vort} = \int d\mu_{vort} e^{iq\oint A} = \mu^2 LT e^{-S_{vort}} e^{iqn2\pi}.$$

Taking the contribution of vortex n and anti-vortex $-n$, with $n > 0$ an integer, we have

$$\left(e^{iq\oint A}\right)_{vort+anti-vort} = 2\mu^2 LT e^{-S_{vort}} \cos(2\pi q n). \tag{3.14}$$

Going back to (3.13) and using that

$$\sum_p \frac{1}{p!}\left(\mu^2 \int d^2x_0 e^{-S_{vort}}\right)^p_{vort+anti-vort} = \exp\left(2\mu^2 LT e^{-S_{vort}}\right),$$

we find, after summing over all p [21]

$$\langle e^{iq\oint A}\rangle = \exp\left(2\mu^2 LT e^{-S_{vort}}\left[\cos(2\pi q n) - 1\right]\right) \quad \Rightarrow \quad V(L) = -2\mu^2 L e^{-S_{vort}}\left[\cos(2\pi q n) - 1\right].$$

In conclusion we found that the potential in the broken phase is confining for $q \notin \mathbb{Z}$, but it is constant for $q \in \mathbb{Z}$. It can be explained by the screening created by ϕ around integer charges [28]. Now the confining for non-integer charges has an exponential decay due to the term $e^{-S_{vort}}$. In a qualitative way the two phases are alike, but in a quantitative analysis they are a little different.

3.2 $D = 3$

Now we will discuss the three dimensional case with one temporal and two spatial dimensions, $\mu = 0, 1, 2$. The first thing we notice is that the potential given in expression (2.38) diverges for $D = 3$, making this potential useless for us here. Next we will see how is the real potential for QED_3 and we will explain why the previous calculation cannot describe this system.

Let us start with the action

$$S = \int d^3x \left(-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right), \tag{3.15}$$

where $[e^2] = 1$ and the photon A_μ has only one propagating polarization state [21]. Given the equation of motion

$$\frac{1}{e^2} \partial_\mu F^{\mu\nu} = J^\nu,$$

and a point charge Q localized at the origin, we can find the potential energy between two charges Q and $-Q$ in the Coulomb gauge ($\nabla \cdot \vec{A} = 0$)

$$\nabla^2 A_0 = Q\delta^2(x) \quad \rightarrow \quad V(r) = -\frac{Q^2}{2\pi} \log\left(\frac{r}{r_0}\right) + cte. \tag{3.16}$$

This is known as the *log confinement* and is shown at the Coulomb phase (that is confining even though is the Coulomb phase), where we have just one massless photon propagating. Let us look into other types of symmetries that are present in the theory.

There is a very important feature in the three dimensional QED. It has an intrinsic $U(1)_{top}$ symmetry associated with the current

$$J^\mu_{top} = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho}, \tag{3.17}$$

which is identically conserved through the Bianchi identity. This is a topological feature of the theory, this is why we have the subscript *top* in the current and in the symmetry $U(1)_{top}$. The conserved charge of this symmetry can be read as the flux of the magnetic field through the spatial surface

$$Q_{top} = \int d^2x J^0_{top} = \frac{1}{2\pi} \int d^2x B. \tag{3.18}$$

So the sources of this flux are magnetic monopoles associated with a local operator $\mathcal{M}(x)$. This means that the monopole operators are charged under the symmetry

$$U(1)_{top}: \quad \mathcal{M} \rightarrow e^{i\alpha} \mathcal{M}. \tag{3.19}$$

The important thing here is that this is not the field A^μ. As this is a topological symmetry it has no realization in the basic field itself, but only on the monopole operators.

Considering now the conservation in the presence of sources

$$\partial_\mu J^\mu_{top} = \delta^3(x). \tag{3.20}$$

Let us find the explicit form of $\mathcal{M}(x)$. First let us look to the partition function in the absence of monopoles

$$Z = \int \mathcal{D}A^\mu \exp\left(-\int d^3x \frac{1}{4e^2} F^{\mu\nu}F_{\mu\nu}\right).$$

As there is no explicit dependence in the field A_μ we can make the integration directly on the field strength $F^{\mu\nu}$. The only thing to be concerned about is the condition $\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = 0$. So we will add in the action a Lagrange multiplier σ, ensuring it,

$$Z = \int \mathcal{D}F_{\mu\nu}\mathcal{D}\sigma \exp\left[-\int d^3x \left(-\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4\pi}\sigma\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}\right)\right]. \tag{3.21}$$

If we change the condition constraint of the field $F^{\mu\nu}$ to the case in the presence of monopoles we would have $\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} - \delta(x) = 0$, so the Lagange multiplier would be $\sigma(\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} - \delta^3(x))$, which is the same as inserting into the partition function the operator $e^{i\sigma(x)}$. Thus we identify the monopole operator as

$$\mathcal{M}(x) \sim e^{i\sigma(x)}, \tag{3.22}$$

with σ being periodic in 2π because of the Lagrange multiplier condition in the presence of sources and the Dirac quantization for magnetic charges.

The next step is to integrate out the fields $F^{\mu\nu}$ to see what is the effective action remaining. Completing the quadratic form in (3.21) and integrating out $F^{\mu\nu}$ we find

$$Z = \exp\left(-\int d^3x \frac{e^2}{8\pi^2}\partial_\mu\sigma\partial^\mu\sigma\right). \tag{3.23}$$

This is the effective action for the system and it has the symmetry

$$U_{top} : \sigma \to \sigma + \alpha, \tag{3.24}$$

as was stated in (3.19). Since this realization of the symmetry is non-linear, the field $\sigma(x)$ can be viewed as the Goldstone boson of the $U(1)_{top}$ broken symmetry. To interpret the field $\sigma(x)$ let us use the Noether theorem in the action (3.23). This leads to

$$J^\mu_{top} = \frac{e^2}{(2\pi)^2}\partial^\mu\sigma. \tag{3.25}$$

Since it need to be the same as the (3.17), we get that

$$\frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} = \frac{e^2}{4\pi^2}\partial^\mu\sigma \quad \to \quad *dA = F_D = d\sigma, \tag{3.26}$$

where we have rewritten the field as $\frac{e^2}{\pi}\sigma \to \sigma$. Bringing back the discussion of the previous chapter we get that the field $\sigma(x)$ is the dual field given by (2.19), but now in $D = 3$ it is a scalar. In other words, the monopole operator $\mathcal{M}(x)$ is no longer an extended 't Hooft operator, but now it is a local object.

Given that the vector field and $\sigma(x)$ have a duality, the Goldstone boson of the broken $U(1)_{top}$ is actually the photon itself. This is a very powerful mechanism used to assure the lack of mass in the photon, because once it is a Goldstone boson, it has no mass by definition.

3.2.1 Abelian Higgs Model

We have described until now the Coulomb phase and how the photon is manifestly massless. Now let us look into the different regimes that the theory can be found using the Abelian Higgs model. Considering a complex scalar field interacting with the potential A^μ we have the action

$$S = \int d^3x \left(-\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + |D_\mu \phi|^2 - m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4 \right). \tag{3.27}$$

The analysis here will be just like in the case of $D = 2$, but now, given that $[e^2] = 1$ the regime of interest is $|m^2| \gg e^4$.

For the case $m^2 \gg e^4$ we can integrate out the scalar field, which leaves us with just the Maxwell theory with the gauge symmetry $U(1)$ unbroken. This is the scenario we discussed above. We have a $U(1)$ gauge symmetry and a topological $U(1)_{top}$ symmetry that is broken, giving us the photon as a Goldstone boson. This is the Coulomb phase, that in a classical regime has a logarithmically confining potential, but in the quantum case we can use the same approach as in the $D = 2$ theory and consider the monopoles operators as instantons, accounting their contribution will make the potential turn into a linear confining [21]. This is why the expression (2.38) cannot describe confinement in $D = 3$, since the calculation did not take into consideration the instanton contributions. So in conclusion, for $m^2 \gg e^4$ we have

$$\text{Coulomb phase} \quad \begin{cases} U(1)_{gauge} & \text{unbroken} \\ U(1)_{top} & \text{spontaneously broken} \end{cases}$$

For $m^2 \ll -e^4$ the gauge invariance is broken, and the scalar field has its vacuum at $|\phi|^2 = -m^2/\lambda$. This is the Higgs phase. Now the photon has mass and the scalar field has vortex excitations (as discussed in Section 3.1). So the same discussion follows here and we have that the winding number is the magnetic flux

$$n = \frac{1}{2\pi i} \oint \frac{\phi^* d\phi}{\phi^* \phi} = \frac{1}{2\pi} \int d^2 B = Q_{top} \in \mathbb{Z}. \tag{3.28}$$

Given that now the topological charge comes from the vortex excitations we can assume that it is the same as the monopole operator we have discussed before. It means that in this regime the charged object under the topological symmetry is the scalar field itself, so the symmetry is not broken. It is in agreement with the fact that now the gauge field is massive and cannot be the Goldstone boson anymore. In sum, we have

$$\text{Higgs phase} \quad \begin{cases} U(1)_{gauge} & \text{spontaneously broken} \\ U(1)_{top} & \text{unbroken} \end{cases}$$

3.3 $D = 4$

The starting point for this section will be the result obtained in (2.38) for $D = 4$

$$V(R) = -\frac{1}{4\pi}\frac{q_e^2}{R}. \tag{3.29}$$

This is the Coulumb potential that characterizes the phase with a massless gauge boson. If we couple the electromagnetic potential to some complex scalar field the analysis will be essentially the same as in the previous sections: for $m^2 > 0$ we have the symmetric phase under $U(1)_{gauge}$ and a massless boson, for $m^2 < 0$ the symmetry is broken and we have a Higgs phase with a massive boson. These two phases can be distinguished by the calculation of Wilson loops, giving first the Coulomb potential and later the constant potential.

The important question here comes from a property of the $D = 3$ case, the topological current. In $D = 4$ we also have the conservation of some topological current coming from the Bianchi identity,

$$J^{\mu\nu}_{top} = \epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}.$$

But this clearly cannot be described by any usual symmetry like we did before. This hints us that we might be missing something, because in the tridimensional case the topological symmetry had a very important role, it gave us the photon as a Goldstone boson. So we might ask whether in $D = 4$ do we also have a topological symmetry leading to a massless boson? The answer is "yes", but now the symmetry is not local as we are used until now. It is a higher form symmetry, and we will talk about it in the next chapter.

4 GENERALIZED HIGHER-FORM SYMMETRY

When we talk about symmetries in QFT the charged objects are mainly local fields $\phi(x)$. They lead to conserved charges, degeneracy in the states, and can be spontaneously broken. In the previous chapter we saw that there are symmetries we can find that does not act on the fields, for example the topological $U(1)_{top}$ symmetry in QED_3 that is realized in terms of basic monopole operators.

Some may ask how such topological symmetries are manifest in other dimensions, and what objects can be charged under it. In this chapter we will see that there is a different kind of symmetry, the so-called $q-$form symmetry [5], which acts on objects $\mathcal{O}(\mathcal{M}^{(q)})$, with $\mathcal{M}^{(q)}$ being a $q-$dimensional manifold. When $q = 0$ we have the usual symmetry. When $q = 1$ we have objects extended along lines, and so forth.

In this chapter and next we will study how these symmetries act and all their consequences. We will also see that they can be spontaneously broken, giving rise to Goldstone bosons.

4.1 Ordinary Symmetries with Differential Forms

Before we start with the higher-form case, let us discuss how usual symmetries can be described with a topological approach. First we will need some useful relations concerning differential forms [19]. Consider a $p-$form Ω_p in a $D-$dimensional manifold

$$\Omega_p = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}, \tag{4.1}$$

with $\wedge$ denoting the anti-symmetric product called wedge product. We can operate on it with an external derivative given by

$$d\Omega_p = \frac{1}{p!}\partial_\alpha\omega_{\mu_1\cdots\mu_p}dx^{\alpha}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}, \tag{4.2}$$

where $d\Omega_p = 0$ when $p = D$ due to the anti-symmetry of the wedge product, and $d\Omega_p$ is a $(p+1)-$form. We can also define the operation of the Hodge dual, given by

$$*\Omega_p = \frac{1}{p!(D-p)!}\omega_{\mu_1\cdots\mu_p}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}dx^{\mu_{p+1}}\wedge\cdots\wedge dx^{\mu_D}, \tag{4.3}$$

which takes a $p-$form to a $(D-p)-$form. The last useful property is the Stokes theorem, that can be stated as

$$\int_{\mathcal{V}} d\Omega_p = \int_{\partial\mathcal{V}}\Omega_p, \tag{4.4}$$

where $\mathcal{V}$ is a $(p+1)-$dimensional volume and $\partial\mathcal{V}$ is a $p-$dimensional closed manifold that encloses $\mathcal{V}$. Now that we have all the important relations let us go to the symmetry in terms of differential forms, which will lead us to a new interpretation of what is a conserved charge.

When we talk about standard symmetries the conserved currents are vectors and tensors. Here we will be interested only on vector or totally anti-symmetric tensor currents. For example, the energy-momentum tensor $T^{\mu\nu}$ is a symmetric object, so it cannot be described by differential forms, since it only works with anti-symmetric components. We can also try the angular momentum tensor $\mathcal{M}^{\mu\nu\rho}$ but it is anti-symmetric only in two of the indices, blocking us to work with it in differential forms too. Given that, let us introduce the concept of a symmetry with a generic $1-$form current

$$J = J_\mu dx^\mu. \tag{4.5}$$

The conservation law for this type of object is classically given by $\partial_\mu J^\mu = 0$, so in order to reproduce it we combine our differential forms properties and get

$$d * J = \partial_\mu J^\mu dV, \tag{4.6}$$

with dV being a $D-$dimensional volume element. So classically the conservation law is stated as $d * J = 0$.

Following the usual procedure with symmetries we can construct a conserved charge integrating our current but we just stated the conservation law with a differential volume, it says that if we are to built a charge, it needs to be defined in a region. Consider a volume Ω with $\mathcal{V} = \partial\Omega$. The conserved charge is

$$Q(\mathcal{V}) = \int_{\mathcal{V}=\partial\Omega} *J. \tag{4.7}$$

Taking the external derivative of $Q(\mathcal{V})$ should lead to the conservation. When the region $\mathcal{V}$ encloses a charged object the conservation law is changed by a source. Using the Stokes theorem (4.4) we find

$$Q(\mathcal{V}) = \int_\Omega d * J = q \int_\Omega \delta^D(x-y) dV = q,$$

and when the volume does not encloses the source it is identically zero. With it we see that this charge is an object strictly dependent of the selected region of interest. This is said to be a topological feature of the charge, because it does not matter if we deform the region, the result only depends if the source is enclosed or not by it. This is a classical conclusion, but in the quantum theory it will also be important.

Now that we have the topological objects, let us compare them with the usual conventions

$$\begin{array}{rcl} \text{Usual notation} & \rightarrow & \text{Differential forms} \\ \partial_\mu J^\mu = 0 & \rightarrow & d * J = 0 \\ Q = \int J^0 dV & \rightarrow & Q(\mathcal{V}) = \int_{\mathcal{V}=\partial\Omega} *J \end{array}$$

The differential form to describe the charge gives us the freedom to choose any space-time surface $\mathcal{V}$ that encloses the charge, as represented in Figure 7. When the surface $\mathcal{V}$ is only

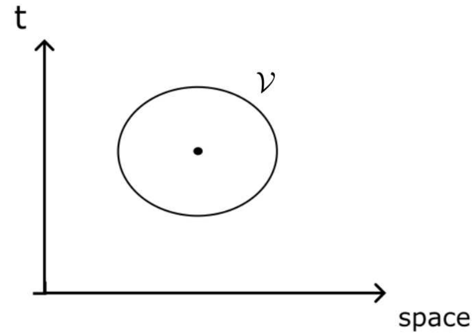

Figura 7 – Charge enclosed by the space-time surface $\mathcal{V}$.

spatial the differential form to describe the charge coincides with the usual one, and it can be interpreted as in the Figure 8. The time interval of integration goes to zero as ϵ vanishes, remaining only a line that is pierced by the charge.

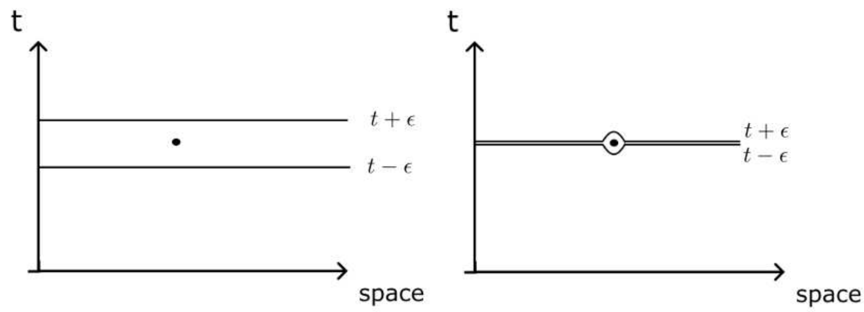

Figura 8 – The usual definition of charge use a fixed time slice. The interception between the charge and the surface is given by taking the time slice to infinite and contracting it with $\epsilon \to 0$.

We can also work with the symmetry in the quantum theory, it just takes us to use the Ward identities instead of the conservation we just saw. Using the result obtained in the Appendix C we have the Ward identity for the field[1]

$$\langle d * J(x)\phi^i(y)\rangle = -i\delta^D(x-y)\,\langle\delta\phi^i(y)\rangle\,. \tag{4.8}$$

Integrating it in a volume Ω we find on the left side

$$\int_\Omega \langle d * J(x)\phi^i(y)\rangle = \int_{\partial\Omega=\mathcal{V}} \langle *J(x)\phi^i(y)\rangle = \langle Q(\mathcal{V})\phi^i(y)\rangle\,,$$

and on the right side

$$\int_\Omega \delta^D(x-y)\,\langle\delta\phi^i(y)\rangle\, d^Dx = link(\mathcal{V},y)\,\langle\delta\phi^i(y)\rangle\,,$$

1 This is the Ward identity under some symmetry that does not show an anomaly in the quantum theory.

where $link(\mathcal{V}, y)$ is the linking number. It measures when the point y is contained in the volume enclosed by $\mathcal{V}$. If $y \in \mathcal{V}$ we have $link(\mathcal{V}, y) = 1$, if $y \notin \mathcal{V}$ the linking number vanishes. So we obtain

$$i \langle Q(\mathcal{V})\phi^i(y)\rangle = link(\mathcal{V}, y) \langle \delta\phi^i(y)\rangle . \tag{4.9}$$

Again it shows the topological aspect of the symmetry, which is explicit in form of a linking number. If the surface encloses the source then the correlation between the charge and source is the variation of the field. But if it does not, there is no correlation between then, making the right side to vanish. The geometrical invariance of the conservation law can be seen in the Figure 9. If we displace Ω by Ω_0 it does not change the linking number, making it a topological feature of the charge.

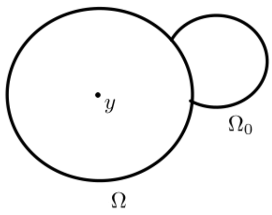

Figura 9 – Consider a little displacement Ω_0 on the region Ω, since the total region $\Omega+\Omega_0$ still contains y the linking number does not change.

Now that we have the charge, it is easy to construct the symmetry operator, with the corresponding parameter of the transformation α,

$$U(\mathcal{V}, \alpha) = e^{i\alpha Q(\mathcal{V})}. \tag{4.10}$$

Considering the variation of the field as $\langle \delta\phi^i(y)\rangle = iM^i{}_j \langle \phi^j(y)\rangle$, with some generator $M^i{}_j$, we can obtain from (4.9) the expression

$$\langle U(\mathcal{V}, \alpha)\phi^i(y)\rangle = R^i{}_j \langle \phi^j(y)\rangle , \tag{4.11}$$

with $R^i{}_j = (e^{i\alpha link(\mathcal{V},y)M})^i{}_j$. This expression tells us that the charged objects are the fields, and they are defined on a $0-$dimensional manifold, that is, the point y. This is said to be a $0-$form symmetry. In the next section we will see how higher-form symmetries work.

4.2 Higher-Form Symmetry

Let us recall the usual symmetry starting with some scalar complex field $\phi(x)$. Using the symmetry operator

$$U_\alpha = e^{i\alpha Q}, \tag{4.12}$$

with Q given by (4.7) and α some infinitesimal parameter, we get the transformation [29]

$$U_\alpha \phi(x) U_\alpha^\dagger = e^{i\alpha q}\phi(x), \tag{4.13}$$

where q is the eigenvalue of the charge operator acting on a charged state (this relation arises from the commutation $\delta\phi(x) = i[Q, \phi(x)] = iq\phi(x)$), that in first order in α is

$$\phi'(x) = \phi(x) + \alpha\delta\phi(x). \tag{4.14}$$

This transformation tells us that the parameter and the charged field share the same nature, both are scalars. Given this, let us look into the dimension of the parameter α, which will give us the higher-form symmetries.

Putting the action on differential forms we see that the Lagrangian needs to be a $D-$form

$$S = \int \mathcal{L},$$

and so will be the integrand on the action variation δS. In the Appendix C we saw that under some local transformation it changes by $\delta S = \int d^D x J^\mu \partial_\mu \xi$ with some parameter ξ. So in the case of our previous symmetry, we can put it in differential forms as

$$\delta S = \int *J \wedge d\alpha, \tag{4.15}$$

where the wedge product sums up the order of the forms that are multiplied. In order to get a $D-$form, since we know that $*J$ is a $(D-1)-$form, we will need $d\alpha$ to be a $1-$form, what is only possible if α is a $0-$form. Recalling (4.14) it makes clearer that the charged field share the same nature as the parameter, both are $0-$form objects.

Now consider the case where the current is a $2-$form. To keep the consistency with the Lagrangian, we need $d\alpha$ to be a $2-$form, so α is a $1-$form. Following the previous analysis we will need to act with the symmetry on a $1-$dimensional object now. It goes like this to $q-$form parameters being responsible for $q-$form symmetries. The important thing here is the parameter, it gives us the order of the higher form symmetry. Let us study in the case of $q = 1$.

4.2.1 $1-$form Symmetry

As we saw before the order of the parameter dictates the order of the higher-form symmetry. Let us analyze the case of a $1-$form parameter. Here we will need a $2-$form current, leading to a charge

$$Q(\Sigma^{(D-2)}) = \int_{\Sigma^{(D-2)}} *J. \tag{4.16}$$

In the $0-$form case we used a symmetry operator defined in a $(D-1)-$dimensional manifold. It allowed us to use the canonical commutation conditions to compute the field variation $\delta\phi = i[Q, \phi(x)]$, since they are defined in a fixed time. In order to use the canonical commutation relations with the symmetry and the charged objects, we need the operator to be defined in a $(D-1)-$dimensional manifold, but now the charge is defined in $(D-2)$ dimensions, so we need to increase the dimension of the integral. The usual way to do it without changing the integral is to put a Dirac delta completing the additional dimension. Here we do not have this option because we are dealing with higher-forms, so we will need an extension of the delta, the Poincare dual.

4.2.1.1 Poincare Dual as the Parameter of Transformation

Considering a closed manifold $\Sigma^{(p-1)} \subset \mathcal{M}^{(p)}$, the Poincare dual of $\Sigma^{(p-1)}$ with support on $\mathcal{M}^{(p)}$ is defined by a $(D-p)-$form given by [30]

$$\xi = \xi_{\mu_{p+1}\cdots\mu_D} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_D}, \tag{4.17}$$

with components

$$\xi_{\mu_{p+1}\cdots\mu_D} = \frac{1}{p!}\int_{\mathcal{M}^{(p)}} \epsilon_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_D}\delta^{(D)}(x-y)dy^{\mu_1}\wedge\cdots\wedge dy^{\mu_p}. \tag{4.18}$$

For the case where $\mathcal{M}^{(p)}$ is just a point x_0, we have $p = 0$, so the Poincare dual is just the Dirac delta valued in the volume $\xi = \delta^{(D)}(x - x_0)dV$. Considering now $p = D - 1$, we have a $1-$form $\xi = \xi_\nu dx^\nu$, with

$$\xi_\nu = \frac{1}{(D-1)!}\int_{\mathcal{M}^{(D-1)}} \epsilon_{\nu\mu_1\cdots\mu_{D-1}}\delta^{(D)}(x-y)dy^{\mu_1}\wedge\cdots\wedge dy^{\mu_{D-1}}. \tag{4.19}$$

Now, considering the region where the charge is defined $\Sigma^{(D-2)} \subset \mathcal{M}^{(D-1)}$, we can extend one dimension in our integral just rewriting it as [30]

$$Q(\Sigma^{(D-2)}) = \int_{\mathcal{M}^{(D-1)}} *J \wedge \xi, \tag{4.20}$$

so that the integral runs over $\mathcal{M}^{(D-1)}$ which allows us to get the canonical commutation between charges and charged objects[2].

The Poincare dual ξ_Σ of a manifold $\Sigma^{(D-2)}$ in $\mathcal{M}^{(D-1)}$ relates both spaces by

$$\int_{\mathcal{M}^{(D-1)}} \eta^{(D-2)} \wedge \xi_\Sigma = \int_{\Sigma^{(D-2)}} \eta^{(D-2)}, \tag{4.21}$$

for every $\eta^{(D-2)}$. Considering that $\Sigma^{(D-2)}$ is a closed spatial manifold, in order to use the canonical quantization condition, we need that ξ has only spatial components too, so we can integrate out the time in the components, finding

$$\begin{aligned}\xi_i &= \frac{1}{(D-1)!}\int_{\mathcal{M}^{(D-1)}} \epsilon_{ij_1\cdots j_{D-2}0}\delta^{(D)}(x-y)dy^{j_1}\wedge\cdots\wedge dy^{j_{D-2}}\wedge dy^0 \\ &= \frac{1}{(D-1)!}\int_{\Sigma^{(D-2)}} \epsilon_{ij_1\cdots\mu_{D-2}}\delta^{(D-1)}(x-y)dy^{j_1}\wedge\cdots\wedge dy^{j_{D-2}},\end{aligned}$$

[2] It is worth to note that in the case of the $0-$form symmetry the Poincare dual was a $0-$form, and using (4.18) with $p = 0$ we find $\xi = 1$. This is why there is no need to act with the parameter on the charge, it is already there.

which implies that $d\xi = 0$, because $\Sigma^{(D-2)}$ has no boundary. This is telling us that the action variation vanishes under transformations of this kind, so they are global unless you open up the manifold where the charge is defined[3].

Going back to the charged objects, let us find their transformation following the description of the 0−form in (4.14). There we have seen that the manifold in which the field was supported was of the same dimension as the one of the parameter, so here we might have the same equivalence between the parameter and the charged operator $W[C]$. Since we have a 1−form parameter it tells us that C is going to be a line. As long as ξ is supported in $\mathcal{M}^{(D-1)}$ and we only need its contribution when it touches the object. We integrate it on the line C and find the transformation

$$W'[C] = W[C] + \int_C \xi(\mathcal{M}^{(D-1)})\delta W[C], \tag{4.22}$$

where the charged objects are 1−form operators. This will be clearer when discussing examples in Chapter 5.

4.2.2 General Description

Now that we have seen how the higher form symmetries arise in the theories, let us put it in a more formal language, generalizing it for p−forms [29, 5, 25, 30, 31]. Until now we have just said that usual symmetries act on local objects and 1−form symmetries act on extended operators, but what are these objects?

In the 0−form case the fields are of the type $e^{i\sigma(x)}$ with $\sigma(x)$ a compact scalar. Making a global transformation is equivalent to map $\sigma'(x) \to \sigma(x)+c$, where c is constant in every point of the spacetime manifold X. For the 1−form case we have as charged objects the Wilson loops introduced in Chapter 2, which are of the form $W(C) = e^{iq\oint_C A}$, with A a 1−form field. Making a global higher form transformation is equivalent to change the field by a constant in the holonomies space, that is, a flat connection $A \to A+\xi$ with $d\xi = 0$, which is the Poincare dual. For a 2−form case we can generalize the concept of a holonomy and now use a 2−form connection B that makes the parallel transport of strings along the space of parameters. It leads to a global transformation of the same form as before $B \to B + \xi_2$, with ξ_2 being the 2−form Poincare dual [30].

This interpretation can be taken to any form with $p \geqslant 0$ dimension, with the charged objects being holonomies of the equivalent p−form fields. Following the previous discussion of (4.15) we know that from such symmetry we have a p−form parameter with a topological current with $(p+1)$ dimension, so the conserved charge is

$$Q(\Sigma^{(D-p-1)}) = \int_{\Sigma^{(D-p-1)}} *J, \tag{4.23}$$

[3] In the reference [30] there is a wider explanation of the relation between manifolds Σ and $\mathcal{M}$ and the global aspect of the transformation.

that using the $p-$form parameter becomes

$$Q(\mathcal{M}^{(D-1)}) = \int_{\mathcal{M}^{(D-1)}} *J \wedge \xi. \tag{4.24}$$

Given this charge we can get the symmetry operators using $U(g, \mathcal{M}) = \exp{(igQ(\mathcal{M}))}$, and the action of them on the charged holonomies is obtained by the Ward identity as we did before. Now let us go to a concrete example, the Maxwell theory.

5 MAXWELL THEORY

The Maxwell theory of electromagnetism is one of the most complete theories we know, but yet we have many things to understand. One of those things are the higher-forms symmetries it has. In this chapter we will unite the topics of Chapters 3 and 4 regarding the topological symmetries. First let us consider the three dimensional case.

5.1 Going back to $D = 3$

When studying the QED_3 we found a topological current given by

$$J^{\mu}_{top} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho}. \tag{5.1}$$

The associated conserved charge is the magnetic flux

$$Q_{top} = \int d^2xB, \tag{5.2}$$

with the charged object given by the monopole operator

$$\mathcal{M}(x) = e^{i\sigma(x)}, \quad *dA = d\sigma. \tag{5.3}$$

Now that we have seen the higher-form mechanism for symmetries we can go back to it. This is the manifestation of a $0-$form symmetry in QED_3. Let us see how to connect both formalism.

First we can check our current in $D = 3$. For this case we have a $1-$form current, which leads to the conservation law $d * J_{top} = 0$ with $*J_{top}$ being a $2-$form, and hence the parameter is a scalar. In the general discussion of symmetries we said that the higher-forms act on holonomies, but here as we are dealing with a $0-$form symmetry, what is the holonomy? Here the "holonomy" will be the scalar monopole operator $\mathcal{M}(x)$ that is dual to the Wilson line. This is called the $0-$form magnetic symmetry.

The important conclusion here is that this magnetic symmetry can be checked on the theory even when we do not know the higher-form mechanism, because it manifests as an usual symmetry. The other conclusion is that the conservation of magnetic flux leads to a magnetic symmetry, so one can ask what happens when we have the conservation of electric flux, namely, the electric symmetry.

The equations of motion in absence of matter are $\partial_{\mu}F^{\mu\nu} = 0$, which leads to a conserved current

$$J^{\mu\nu}_{e} = F^{\mu\nu}, \tag{5.4}$$

with the conservation law $d * J_e = 0$. Since this current is a 2−form we have a 1−form symmetry, where the charge is

$$Q(\mathcal{V}^{(1)}) = \int_{\mathcal{V}^{(1)}} *J_e = \frac{1}{2}\int_{\mathcal{V}^{(1)}} \epsilon_{\mu\nu\rho}F^{\nu\rho}dx^{\mu}. \tag{5.5}$$

Considering a spatial line at fixed time $\mathcal{V}^{(1)}$ we find

$$Q(\mathcal{V}^{(1)}) = \frac{1}{2}\int_{\mathcal{V}^{(1)}} \epsilon_{ij0}F^{j0}dx^{i} = \int_{\mathcal{V}^{(1)}} \epsilon_{ji}E^{j}dx^{i}, \tag{5.6}$$

that is the flux of the electric field across the line. Considering the Wilson line valued in a spatial curve C as the charged object, its transformation is

$$\begin{aligned} W'[C] &= U(\alpha, \mathcal{V}^{(1)})W[C]U^{\dagger}(\alpha, \mathcal{V}^{(1)}) \\ &= \exp\left(i\alpha\int_{\mathcal{V}^{(1)}} \epsilon_{ji}E^{j}dy^{i}\right)\exp\left(iq\int_{C} A_i dx^{i}\right)\exp\left(-i\alpha\int_{\mathcal{V}^{(1)}} \epsilon_{ji}E^{j}dy^{i}\right). \end{aligned}$$

This is where the canonical quantization takes place, once C is a spatial curve we can use the canonical quantization condition for the electromagnetic field

$$[A^{i}(x), E^{j}(y)] = -i\delta^{ij}\delta^{2}(x-y). \tag{5.7}$$

Here we will need the Poincare dual to match the dimensions. So considering that this is a 1−form symmetry, we will need a 1−form parameter $\xi = \xi_i dx^i$, which extends our integral to fulfill the whole $\mathbb{R}^2$, leading to the transformation

$$\begin{aligned} W'[C] &= \exp\left(i\alpha\int_{\mathbb{R}^2} \epsilon_{ji}E^{j}\xi_n dy^{i}\wedge dy^{n}\right)\exp\left(iq\int_{C} A_i dx^{i}\right)\exp\left(-i\alpha\int_{\mathbb{R}^2} \epsilon_{ji}E^{j}\xi_m dy^{i}\wedge dy^{m}\right) \\ &= \exp\left(-i\alpha\int_{\mathbb{R}^2} \epsilon_{ij}E^{j}\xi_n\epsilon^{in}dS_y\right)\exp\left(iq\int_{C} A_i dx^{i}\right)\exp\left(i\alpha\int_{\mathbb{R}^2} \epsilon_{ij}E^{j}\xi_m\epsilon^{im}dS_y\right), \end{aligned}$$

where $dy^j \wedge dy^n = \epsilon^{jn}dS_y$ is the volume in two dimensions. Using the expression (5.7), we find the useful commutator

$$\begin{aligned} \left[iq\int_{C} A_i dx^{i}, i\alpha\int_{\mathbb{R}^2} E_m\xi^{m}dS_y\right] &= -\alpha q\int_{C}\int_{\mathbb{R}^2}[A_i(x), E^{m}(y)]\xi_m dx^{i}dS_y \\ &= i\alpha q\int_{C}\int_{\mathbb{R}^2} \delta^{2}(x-y)\xi_m dx^{m}dS_y \\ &= i\alpha q\int_{C} \xi_m dx^{m} = i\alpha q\int_{C} \xi. \end{aligned}$$

Putting it back into the Wilson line transformation and using the BCH formula $e^{A}e^{B} = e^{B}e^{A}e^{[A,B]}$, we find[1]

$$W'[C] = e^{i\alpha q\int_C \xi}W[C], \tag{5.8}$$

that when expanded in first order in α recovers (4.22). It is worth to check that if we put the explicit form of the Wilson line we can see that the field changes by $A \to A + \alpha\xi$, that is exactly what we wanted, a shift under a flat connection.

[1] We could have done the calculation without the addition of the Poincare dual, we would find the same result, as expected, since it does not change the physics of the system.

Now let us search for some topological features by opening the Poincare dual in the integral. Here we are dealing with only the spatial dimension, so we will use the expression (4.19) with $D = 2$. It leads us to

$$\int_C \xi_i dx^i = \int_{C_x} \int_{\mathcal{V}_y^{(1)}} \epsilon_{ij} \delta^2(x-y) dx^i dy^j, \tag{5.9}$$

that is, the Poincare dual is constructed with support on the manifold where the charge is localized, and the transformation of the Wilson line only occurs when there is an interception between the charge and the line C. This is called the interception number, and it can be viewed in Figure 10.

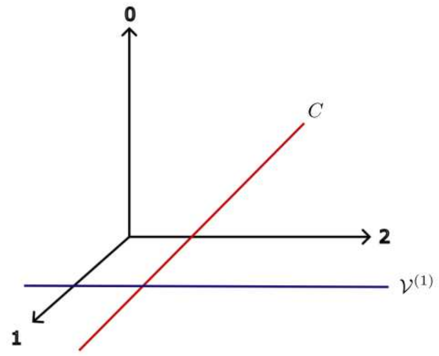

Figura 10 – Intersection between the line $W(C)$ and the charge supported on the line $\mathcal{V}^{(1)}$.

Now let us go to the functional formalism, where we can approach the case of curves going through space and time. We are going to deduce the Ward identity by considering $\langle W[C]\rangle = \langle W'[C]\rangle$. Using

$$W'[C] = W[C] + \alpha \int_C \xi \delta W[C], \tag{5.10}$$

and the action transformation (4.15) we get

$$\langle W'[C]\rangle = \int \mathcal{D}A \left(W[C] + \alpha \int_C \xi \delta W[C] \right) \left(1 + i\alpha \int *J \wedge d\xi \right) e^{iS[A]} = \langle W[C]\rangle .$$

Keeping only terms of first order in α we get

$$i \int \langle *J \wedge d\xi W[C]\rangle + \int_C \xi \langle \delta W[C]\rangle = 0. \tag{5.11}$$

Considering the first integral running through the space-time volume $\mathcal{V}$ we can integrate by parts and find

$$i \int_{\mathcal{V}} \langle d * J \wedge \xi W[C]\rangle = \int_C \xi \langle \delta W[C]\rangle , \tag{5.12}$$

where $\mathcal{V}$ is a tridimensional space-time manifold. Recalling (4.21) we can put this integral in a two dimensional manifold Ω with $\partial\Omega = S$, where S is the line where the charge is defined. This gives

$$i \int_{\Omega} \langle d * JW[C]\rangle = \int_C \xi \langle \delta W[C]\rangle . \tag{5.13}$$

This is the Ward identity for the 1−form symmetry of the QED_3. We can take a step further and use the Stokes theorem to find the topological charge

$$i\left\langle QW[C]\right\rangle = \int_C \xi \left\langle \delta W[C]\right\rangle, \tag{5.14}$$

that being exponentiated, and using $\delta W[C] = iqW[C]$, gives

$$\left\langle U(\alpha, S)W[C]\right\rangle = e^{iq\alpha\int_C \xi}\left\langle W[C]\right\rangle. \tag{5.15}$$

Let us come back to expression (5.13), where we have used the Poincare dual valued in the surface Ω. If we use the definition of ξ in Ω in the right side of this equality we get

$$\int_C \xi_\mu dx^\mu = \int_C\int_\Omega \epsilon_{\mu\nu\sigma}\delta^3(x-y)dx^\mu \wedge dy^\nu \wedge dy^\sigma = \int_C\int_\Omega \delta^3(x-y)dx^\mu d\Omega_\mu. \tag{5.16}$$

If we integrate out the time component of dy^0 we go back to (5.9). It tells us that the action of the symmetry is non-trivial when there is interception between the surface Ω and the line C. This is the same as the linking number and is depicted in the Figure 11.

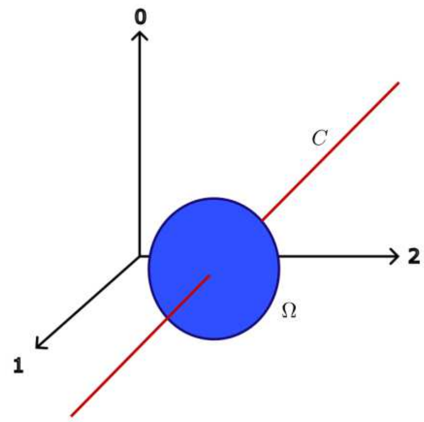

Figura 11 – Link between line $W(C)$ and the surface Ω enclosed by the line S.

The general conclusion of the equation (5.15) is the topological meaning of the theory. Now we have the same interpretation as in the 0−form case. If the charged object pierces the region of the charge, then it feels the transformation. There we had a point piercing a volume, because it was a 0−form in D dimensions. Here in a 1−form symmetry in $3d$ we have a line piercing a surface, which is invariant under geometrical changes in Ω.

5.2 Maxwell in $D = 4$

Up to this point we have introduced the formalism of higher-form symmetries and used as an example the QED_3. Now we are going to apply it on the four dimensional case.

In the absence of electric and magnetic charges, the Maxwell equations are

$$dF = 0 \quad \text{and} \quad d * F = 0, \tag{5.17}$$

with the first being the Bianchi identity and the second the equation of motion of the vector field. Following the previous case we can use these equations to define the electric and magnetic currents

$$J_e = F \quad \text{and} \quad J_m = *F, \tag{5.18}$$

with the conservation law $d * J = 0$. In four dimensions these currents are both $2-$form objects, so here we will have two $1-$form symmetries given by the charge generators

$$Q_e(S) = \int_S *J_e = \int_S *F, \tag{5.19}$$

$$Q_m(S') = \int_{S'} *J_m = \int_{S'} F, \tag{5.20}$$

where S and S' are two dimensional surfaces.

Given the intuition we got in the QED_3 we already know that the charged object under the $1-$form electric symmetry is the Wilson line, and the one for the magnetic symmetry is its dual, the monopole operator, which in four dimensions is the 't Hooft line. Here we will be approaching the symmetries first with the canonical quantization, using the commutation relations. Later we will use the path integral formalism to better visualize the topological meaning of the symmetries.

5.2.1 Canonical Quantization Approach

Once again we need to obtain the action of the symmetry operator on the charged object. Let us start with the electric symmetry. Considering an infinite spatial surface S we have the charge

$$\begin{aligned} Q_e(S) &= \int_S \frac{1}{2}(*F)_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= \frac{1}{4}\int_S \epsilon_{ijk0} F^{k0} dx^i \wedge dx^j \\ &= -\int_S E_k dS^k, \end{aligned}$$

which is the electric flux through S. With it we have the symmetry operator

$$U(\alpha, S) = \exp\left(-i\alpha \int_S E_k dS^k\right), \tag{5.21}$$

that when applied to the spatial Wilson line valued in C, leads to

$$W'[C] = \exp\left(-i\alpha \int_S E_k dS^k\right) \exp\left(iq_e \int_C A_i dx^i\right) \exp\left(i\alpha \int_S E_k dS^k\right).$$

We can approach it here in the same manner as before, using the canonical commutation relation

$$[A^i(x), E^j(y)] = -i\delta^{ij}\delta^3(x-y), \tag{5.22}$$

that leads to the intersection between the line C and the surface S

$$\begin{aligned}\left[iq_e \int_C A_i dx^i, i\alpha \int_S E_k dS_y^k\right] &= iq_e\alpha \int_C \int_S \delta^3(x-y) dx_i dS_y^i \\ &= iq_e\alpha \int_C \xi_i dx^i,\end{aligned} \tag{5.23}$$

where we have used

$$\xi_i = \frac{1}{2}\int_S \epsilon_{ijk}\delta^3(x-y)dy^j \wedge dy^k. \tag{5.24}$$

Again we find that the line changes as

$$W'[C] = e^{iq_e\alpha \int_C \xi} W[C], \tag{5.25}$$

where the intersection can be viewed in the Figure 12.

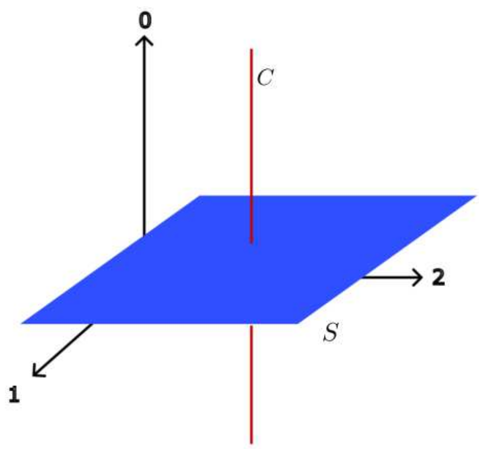

Figura 12 – Intersection between line $W(C)$ and the surface S where the charge is defined.

Now let us go to the magnetic symmetry in the canonical quantization scheme. First we need to calculate the charge valued in a spatial surface S. We have

$$\begin{aligned}Q_m(S) &= \frac{1}{2}\int_S F_{ij}dx^i \wedge dx^j \\ &= -\int_S \epsilon_{ijk}B^k dx^i \wedge dx^j \\ &= -\int_S B_k dS^k.\end{aligned}$$

This is the magnetic flux through S, and the symmetry operator is

$$U(\alpha, S) = \exp\left(-i\alpha \int_S B_k dS^k\right). \tag{5.26}$$

As we saw in the electric case the symmetry acts on lines constructed with its canonical conjugate, which were the field $A^i(x)$ and their momentum $E^j(y)$. So here we need to find the field that has $B^i(x)$ as its momentum.

First let us write the Maxwell action in terms of the dual field strength (the dual action for D dimensions can be found at Appendix E)

$$S = -\frac{1}{4}\int d^4x(*F)^{\mu\nu}(*F)_{\mu\nu} = -\frac{1}{4}\int d^4x(\partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu)^2, \tag{5.27}$$

where $*F = *dA = d\bar{A}$. The canonical momentum from this action is

$$\bar{\Pi}^i(y) = \frac{\partial \mathcal{L}}{\partial \dot{\bar{A}}^i} = -B^i. \tag{5.28}$$

So the canonical commutation relation is

$$[\bar{A}^i(x), B^j(y)] = -i\delta^{ij}\delta^3(x-y), \tag{5.29}$$

leading us to conclude that the charged object is the 't Hooft line

$$T[C] = \exp\left(iq_m \int_C \bar{A}\right). \tag{5.30}$$

This is all we need to analyze the action of the symmetry operator on the $T[C]$. Following the electric case calculation, we get

$$\begin{aligned} T'[C] &= U(\alpha, S)T[C]U^\dagger(\alpha, S) \\ &= e^{i\alpha q_m \int_C \xi} T[C], \end{aligned} \tag{5.31}$$

with ξ given by (5.24). The discussion of the intersection number here is the same as in the electric case, but now we have the dual vector field and the magnetic field.

5.2.2 Path Integral Approach

With the path integral formalism we can treat general defects $W[C]$, with C being a space-time line. For this case we will need to find the Ward identity and then the action of the symmetry operator. For simplicity we will only calculate it for the electric symmetry. The one for the magnetic symmetry follows straightforwardly.

In the $3d$ case we have used the charge with support on the Poincare dual. It allowed us to calculate the action of the symmetry on the line in the canonical and path integral formalism. Now in the 4d case we choose to start the canonical derivation without ξ and in the final we found it, proving that it is something that can be neglected in the charge and yet lead to the same result. Now we will do the same thing in the path integral formalism. We start with the charge without the dual and later find it in the linking number. This allows us to see the theory from a different perspective.

One alternative way to find the Ward identity is using the Schwinger-Dyson equation [32]

$$(\Box_x + m^2) \langle \phi_x \mathcal{O}(\phi) \rangle = \langle \mathcal{L}'_{int}[\phi_x] \mathcal{O}(\phi) \rangle - i \frac{\delta}{\delta \phi_x} \langle \mathcal{O}(\phi) \rangle ,$$

for some operator $\mathcal{O}(\phi) = \phi_1 \cdots \phi_n$. Using that $\frac{\delta S}{\delta \phi_x} = (\Box_x + m^2)\phi_x - \mathcal{L}'_{int}[\phi_x]$ we have

$$\langle \frac{\delta S}{\delta \phi_x} \mathcal{O}(\phi) \rangle = -i \frac{\delta}{\delta \phi_x} \langle \mathcal{O}(\phi) \rangle . \tag{5.32}$$

Now we can substitute ϕ for A^μ and the operator $\mathcal{O}(\phi)$ for $W[C]$. With this we find

$$\langle \frac{\delta S}{\delta A_\mu} W[C] \rangle = -i \frac{\delta}{\delta A_\mu} \langle W[C] \rangle . \tag{5.33}$$

Giving that the action is $S \sim F_{\mu\nu} F^{\mu\nu}$ we get

$$\langle \partial_\nu F^{\nu\mu}(x) W[C] \rangle = -i \frac{\delta}{\delta A_\mu} \langle W[C] \rangle , \tag{5.34}$$

and the derivative of the Wilson line is

$$\frac{\delta W[C]}{\delta A_\mu(x)} = i q_e \int_C \delta^4(x-y) dy^\mu W[C].$$

Substituting in the previous expression we get

$$\langle \partial_\nu F^{\nu\mu}(x) W[C] \rangle = q_e \int_C \delta^4(x-y) dy^\mu \langle W[C] \rangle . \tag{5.35}$$

The next step is to integrate both sides in an oriented volume $\mathcal{V}$ that contains the topological charge

$$\int_{\mathcal{V}} \epsilon_{\mu\gamma\sigma\rho} \langle \partial_\nu F^{\nu\mu}(x) W[C] \rangle dV_x^{\gamma\sigma\rho} = q_e \int_{\mathcal{V}} \int_C \epsilon_{\mu\gamma\sigma\rho} \delta^4(x-y) dy^\mu dV_x^{\gamma\sigma\rho} \langle W[C] \rangle .$$

Let us study each side of this equality. First we consider the left side (LS). Using that $F^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} (*F)_{\alpha\beta}$ and $dV_x^{\gamma\sigma\rho} = \epsilon^{\gamma\sigma\rho\lambda} \eta_\lambda dV_x$, with η_λ being an unitary vector that indicates the direction of the manifold, we have

$$\begin{aligned} LS &= \int_{\mathcal{V}} \langle \partial_\nu (*F)_{\alpha\beta}(x) W[C] \rangle \epsilon^{\mu\nu\alpha\beta} \underbrace{\epsilon_{\mu\gamma\sigma\rho} \epsilon^{\gamma\sigma\rho\lambda}}_{-\delta^\lambda_\mu} \eta_\lambda dV_x \\ &= - \int_{\mathcal{V}} \langle \partial_\nu (*F)_{\alpha\beta}(x) W[C] \rangle \epsilon^{\lambda\nu\alpha\beta} \eta_\lambda dV_x \\ &= \int_{\mathcal{V}} \langle \partial_\nu (*F)_{\alpha\beta}(x) W[C] \rangle dV_x^{\nu\alpha\beta} = \int_{\mathcal{V}} \langle d * F W[C] \rangle . \end{aligned}$$

Using the Stokes theorem and $\Omega = \partial \mathcal{V}$ being the surface that encloses the charge, we have

$$LS = \int_\Omega \langle *J W[C] \rangle = \langle Q_e(\Omega) W[C] \rangle . \tag{5.36}$$

In the right side (RS) we have exactly the Poincare dual with support on the tridimensioal volume $\mathcal{V}$

$$RS = q_e \int_C \underbrace{\int_{\mathcal{V}} \epsilon_{\mu\gamma\sigma\rho}\delta^4(x-y)dV_x^{\gamma\sigma\rho}}_{\xi_\mu} dy^\mu \langle W[C]\rangle = q_e \int_C \xi \langle W[C]\rangle .$$

So the complete expression reads

$$\langle Q_e(\Omega)W[C]\rangle = q_e \int_C \xi \langle W[C]\rangle . \tag{5.37}$$

Multiplying both sides by $i\alpha$ and exponentiating leads to the finite transformation

$$\langle U(\alpha,\Omega)W[C]\rangle = e^{i\alpha q_e \int_C \xi} \langle W[C]\rangle . \tag{5.38}$$

Now that we have the action of the symmetry on the line let us discuss the topological interpretation using the linking number given in (5.37). In the tridimensional case we had the intersection between the Wilson line and the line where the charge was defined, when we added the time using path integral the intersection became the link between the line and a surface that was enclosed by the manifold of the charge. Here we have the same. In the canonical quantization the intersection was between the line and the plane where the charge was defined, but now in the path integral the time comes in and add one more dimension, making the link to be between the line and the volume that is enclosed by the charge surface. Now we cannot see the time dimension since the volume already exhausts the three dimensions we see, but we can try with some illustration. In the Figure 13 we see the closed spatial volume $\mathcal{V}$ and the line C needs to be orthogonal to the volume, that is, it is only temporal, so all we can see is the link between the line and the volume in an instant of time, that is, all we see is a red point centered in the ball. If we could picture the fourth dimension we would see the continuity of temporal dots forming the line C.

5.3 General Remarks on Maxwell Higher-Forms

We have just described the two 1−form symmetries present in QED_4, the electric and magnetic. In the tridimensional case we saw that the symmetries were a 1−form electric and a 0−form magnetic, so now we can put it in terms of the dimensions. Given the currents

$$J_e = F \quad \text{and} \quad J_m = *F, \tag{5.39}$$

we see that it does not matter the dimension, we will always have a 2−form electric current, so there is always a 1−form electric symmetry denoted by $U(1)_e^{(1)}$. However the magnetic case depends on the dimension because the current is a $(D-2)$−form, which leads to a $(D-3)$−form symmetry $U(1)_m^{(D-3)}$.

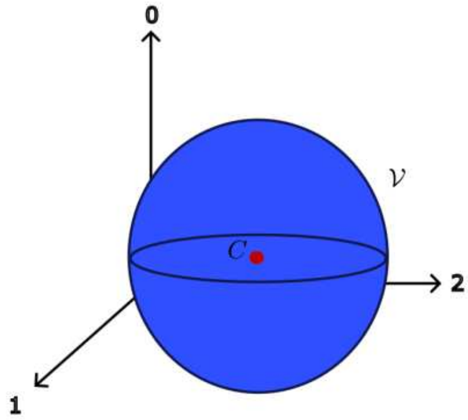

Figura 13 – Link between the temporal line $W(C)$ and the volume $\mathcal{V}$ enclosed by the surface Ω.

When we found the $U(1)_m$ symmetry in the QED_3 we also studied the spontaneous breaking of it, which lead us to two different phases, where the dual photon was the Goldstone boson in the broken phase. Since we also have a magnetic $U(1)$ symmetry now, it is intuitive that we can make the same discussion here, but now the symmetry is a $1-$form, so the mechanism used does not suit this case. In order to see which phases show up with the $U(1)^{(1)}_m$ magnetic symmetry in $D = 4$ we need to introduce the SSB for higher-forms, that is the topic of the next chapter.

6 SSB FOR P-FORMS

In the previous chapters we presented a new type of symmetry, the higher-form symmetry. When applying it to the electromagnetism in 4 dimensions we had the electric and magnetic symmetries with the charged objects being the Wilson and 't Hooft lines, respectively. This type of symmetry can be reinterpreted in terms of topology, and we have seen that for the 3 dimensional QED that when there is such topological symmetry its spontaneous breaking leads to the dual photon as a Goldstone boson. For the 4 dimensional case we will have something alike, this is what we will see in this chapter.

6.1 Order Parameter for Higher-Forms

For the spontaneous breaking of a $0-$form symmetry with charge Q and a charged object ϕ the local order parameter is given by

$$O(x) = \langle \delta\phi(x) \rangle = \langle [Q, \phi(x)] \rangle \,, \tag{6.1}$$

which comes from the symmetry condition $\langle \phi' \rangle = \langle \phi \rangle$ for symmetric states and $\langle \phi' \rangle \neq \langle \phi \rangle$ for spontaneously broken states.

Since the higher-form symmetry is also a symmetry, it can be spontaneously broken and lead to different phases in the theory. In order to describe this mechanism the same logic can be used to define an order parameter, but now we use the line operators as charged objects. Considering the electric symmetry, let us use the Wilson loop as an example and impose $\langle W'(C) \rangle = \langle W(C) \rangle$ for symmetric states and $\langle W'(C) \rangle \neq \langle W(C) \rangle$ for spontaneously broken states. This leads us to consider the object $\langle \delta W(C) \rangle$ as an order parameter. If it is vanishing then we are in the symmetric phase, otherwise we are in the broken phase.

Starting with the canonical quantization we have the transformation

$$W'[C] = U(\alpha, \mathcal{V}^{(1)}) W[C] U^{\dagger}(\alpha, \mathcal{V}^{(1)}). \tag{6.2}$$

Thus using $U(\alpha, \mathcal{V}^{(1)}) = \exp(i\alpha Q)$ we have $\delta W(C) \sim [Q, W(C)]$, so the order parameter will be

$$O(C) = \langle \delta W(C) \rangle = \langle [Q, W(C)] \rangle \,. \tag{6.3}$$

It is worth to note that this is not an usual order parameter, it is not local. This seems to go against the Landau paradigm which says that a phase transition must be described by a local order parameter. Here we have an extended object and we are dealing with topological phases, so there is no need to expect the same behavior for the order parameter as we have for usual phases.

Recalling the previous chapter we have seen that the charge for the electric symmetry and the Wilson loop are built up with canonical conjugates, so that the transformation is

$$W'(C) = e^{i\alpha q \int_C \xi} W(C) \approx W(C) + i\alpha q \int_C \xi W(C) + \mathcal{O}(\alpha^2), \tag{6.4}$$

which leads to

$$O_e(C) = \langle \delta W(C) \rangle \sim \langle W(C) \rangle . \tag{6.5}$$

This is the final form of the order parameter for the electric symmetry. In the case of the magnetic one we could make the same construction but with the 't Hooft loop

$$O_m(C) = \langle T(C) \rangle . \tag{6.6}$$

In Section 2.3 we discussed the general calculation of $\langle W(C) \rangle$ which will be very useful here. There we saw that there are three possibilities for the decay of the line, the area, perimeter and Coulomb law. We can connect these three possible behaviors with the order parameter by taking the limit of a large curve C. Comparing the limit of the three possible decays we have

$$\langle W(C) \rangle = \begin{cases} e^{-A(C)} \sim 0 \\ e^{-P(C)} \neq 0 \\ Coulomb \neq 0 \end{cases} \tag{6.7}$$

The area law vanishes faster than the other two, so the confining behavior is equivalent to consider the symmetric phase, and the other two are the ones for the spontaneously broken phase.

6.2 Goldstone Boson for Electric 1-form Symmetry

Now we are going to show how to obtain the Goldstone boson of QED_4 from the SSB of a 1-form symmetry. First we need to recall the Ward identity for the Wilson line (5.13). Removing the integral in Ω and using $\langle \delta W(C) \rangle = iq_e W(C)$, with vector notation we find

$$\langle \partial_\mu J^{\mu\nu}(x) W(C) \rangle = q_e \oint_C dy^\nu \delta^4(x-y) \langle W(C) \rangle . \tag{6.8}$$

This is directly proportional to the order parameter we introduced in the previous section. Let us take the Fourier transform of this expression. In the left side we find

$$\int d^D x e^{ipx} \langle \partial_\mu J^{\mu\nu}(x) W(C) \rangle = -i \int d^D x e^{ipx} p_\mu \langle J^{\mu\nu}(x) W(C) \rangle . \tag{6.9}$$

On the right side we have

$$q_e \int d^D x e^{ipx} \oint_C dy^\nu \delta^4(x-y) \langle W(C) \rangle = q_e \oint_C dy^\nu e^{ipy} \langle W(C) \rangle . \tag{6.10}$$

Combining both sides we get

$$p_\mu \langle J^{\mu\nu}(p) W(C) \rangle = i q_e f^\nu(p,C) \langle W(C) \rangle , \quad \text{with} \quad f^\nu(p,C) \equiv \oint_C dy^\nu e^{ipy},$$

where for a closed path $p_\nu f^\nu(p,C) = 0$ and $f^\nu(0,C) \neq 0$. With all this we conclude that

$$\langle J^{\mu\nu}(p) W(C) \rangle \sim \frac{p^\mu f^\nu(p,C) - p^\nu f^\mu(p,C)}{p^2}. \tag{6.11}$$

It tells us that there is a pole in $p^2 = 0$ in the correlation, so there must be some gapless mode in the theory. When we have studied the SSB formalism in Chapter 1 we have seen that the Goldstone boson will always be the one with the non-linear realization of the symmetry. In that case we had $\lambda' \to \lambda + const$. In the 1−form symmetry case we have seen that the field who has the transformation of this kind is the gauge field itself, which changes by $A \to A + \xi$, so this is going to be the Goldstone boson of this theory.

Now it is clear why the field changes in this manner, this is because the symmetry is spontaneously broken and has a non-linear realization. The photon given by the field A has all the properties of a Goldstone boson, namely it has no mass and propagate in the vacuum of the theory. We can also check (by canonical quantization) that the state excited by the conserved current is the same as the one photon state. First let us define the state of one photon with momentum p and polarization λ as

$$|\lambda, p\rangle = A^\dagger_\lambda(p) |0\rangle , \tag{6.12}$$

where $A^\dagger_\lambda(p)$ comes from the canonical quantization of the photon

$$A_\mu(\vec{x}) = \int \frac{d^3p}{2|p|} \sum_{\lambda=1}^{4} \epsilon^\lambda_\mu(p) \left[A_\lambda(p) e^{ipx} + A^\dagger_\lambda(p) e^{-ipx} \right] . \tag{6.13}$$

Giving the equation of motion for the field and the Gupta-Bleuler condition for the physical states, we eliminate two degrees of freedom from the field, remaining only the polarizations $\lambda = 1, 2$. These are the physical polarizations that lead to the quantization condition

$$[A_\lambda(p), A^\dagger_{\lambda'}(p')] = 2|p| \delta_{\lambda\lambda'} \delta^3(p - p'). \tag{6.14}$$

The next step is to check whether the Goldstone state $f^{\mu\nu} |0\rangle$ has projection on other states or not. First we can try on the vacuum, it gives us

$$\langle 0 | f^{\mu\nu} | 0 \rangle = 0, \tag{6.15}$$

because $f^{\mu\nu}$ depends linearly on the operators $A_\lambda(p)$ and $A^\dagger_\lambda(p)$. Next we try on the one photon state and find

$$\begin{aligned}
\langle 0|f^{\mu\nu}|\lambda,p\rangle &= \langle 0|(\partial^\mu A^\nu - \partial^\nu A^\mu)A^\dagger_\lambda(p)|0\rangle \\
&= \langle 0|i\int \frac{d^3p'}{2|p'|}\sum_{\lambda=1,2}\left(\epsilon^\nu_{\lambda'}(p')p'^\mu - \epsilon^\mu_{\lambda'}(p')p'^\nu\right)A_{\lambda'}(p')e^{-ip'x}A^\dagger_\lambda(p)|0\rangle \\
&= ie^{-ipx}\,\langle 0|\epsilon^\nu_\lambda(p)p^\mu - \epsilon^\mu_\lambda(p)p^\nu|0\rangle \neq 0.
\end{aligned}$$

Thus we conclude that there is a non-vanishing projection of the Goldstone boson state on the one photon state. It is simple to see that there is no projection of the Goldstone state onto any state with two or more photons. It tells us that the Goldstone state is exactly the one photon state.

6.3 Generalization for p-form Symmetries

The mechanism of spontaneous breaking of a symmetry can be extended to p-form symmetries. Here we will do the same discussion as in Section 1.3, but now with the generalized charge

$$Q(M^{(D-p-1)}) = \int_{\Sigma^{(D-1)}} *J\wedge\xi.$$

Considering that we are dealing with fixed time, the charge $Q(M^{(D-p-1)})$ is a 0-form [30], and can be written as

$$Q(M^{(D-p-1)}) = \int_{\Sigma^{(D-1)}} (*J\wedge\xi)(x)d^{D-1}x.$$

So, using $W(C)$as the charged object, with C being a p- manifold, the order parameter will be

$$\mathcal{O} = \langle 0|[Q(M),W(C)]|0\rangle = \int_\Sigma d^{D-1}x\,\langle 0|[(*J\wedge\xi)(x),W(C)]|0\rangle\,. \tag{6.16}$$

At this point we can introduce a set of energy eigenstates $\mathbb{I} = \sum_n |n\rangle\langle n|$. Which leads to

$$\begin{aligned}
\mathcal{O} &= \int_\Sigma d^{D-1}x\,(\langle 0|(*J\wedge\xi)(x)W(C)|0\rangle - \langle 0|W(C)(*J\wedge\xi)(x)|0\rangle) \\
&= \sum_n\int_\Sigma d^{D-1}x\,(\langle 0|(*J\wedge\xi)(x)\,|n\rangle\langle n|\,W(C)|0\rangle - \langle 0|W(C)\,|n\rangle\langle n|\,(*J\wedge\xi)(x)|0\rangle)\,.
\end{aligned}$$

Using the time and space translation operators we find

$$\mathcal{O} = \sum_n(2\pi)^{D-1}\delta^{D-1}(\vec{p})\left(\langle 0|(*J\wedge\xi)(0)\,|n\rangle\langle n|\,W(C)|0\rangle\,e^{-i\omega_n t} - \langle 0|W(C)\,|n\rangle\langle n|\,(*J\wedge\xi)(0)|0\rangle\,e^{i\omega_n t}\right).$$

Giving this expression we can see that when the symmetry is spontaneously broken $\mathcal{O}\neq 0$, there is an energy state created by the current. This is the same scenario we had in the 0- form case.

We can also use the time conservation of the order parameter and find

$$\partial_t \mathcal{O} = -i \sum_n (2\pi)^{D-1} \omega_n \delta^{D-1}(\vec{p}) \Big(\langle 0|(*J \wedge \xi)(0) |n\rangle \langle n| W(C)|0\rangle e^{-i\omega_n t} + \\ + \langle 0|W(C) |n\rangle \langle n| (*J \wedge \xi)(0)|0\rangle e^{i\omega_n t} \Big) = 0.$$

That is, when the wave number goes to zero the energy vanishes. This is the condition for the Goldstone boson to be massless.

CONCLUSION

Along this work we have studied several aspects of higher-form symmetries. The Bianchi identity led us to a topological symmetry in $D = 3$, which resulted in the dual photon as a Goldstone boson. In $D = 4$ the Bianchi identity still leads to a conservation law, not of an ordinary current, but instead of a higher-order current. This analysis motivated us to consider a new type of symmetry, the higher-form one. We have seen how to construct topological charges and discussed their physical interpretation. The key property is the identification of conserved charges and the charged operators as geometrical objects that possess nontrivial link. In this sense, the charge operator of an ordinary symmetry in $D = 4$ is defined in a S^3 manifold that links with a point, which in turn is associated with a local operator. For a line operator, the charge is defined on a S^2 manifold, which has a nontrivial link with the line. As in the usual case, the higher-form symmetry can also be spontaneously broken. In this way, the photon of QED_4 is reinterpreted as the Goldstone boson of electric and magnetic 1-form symmetries.

The implementation of the 1-form symmetry in the electromagnetism can also be extended to non-Abelian case. For example, a $SU(N)$ gauge theory has a discrete $\mathbb{Z}_N$ center 1-form symmetry, which acts on the Wilson lines in the fundamental representation [5, 33]. Higher-form symmetries involving orthogonal gauge groups have also been considered in [34] and possess interesting potential applications in condensed matter [35].

Generalized symmetries constitute a whole new branch of physics, with far-reaching consequences. As we have seen with the photon being the Goldstone boson, these new types of symmetries lead us to a deeper understanding of the structure of physical theories and to surprisingly new insights. The following years will certainly be full of new discoveries along these directions, and I hope to take part of it.

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A WILSON LOOP CALCULATION WITH THE PROPAGATOR

In this appendix we are going to start from the expression in Euclidian space-time

$$\langle W(\Gamma)\rangle = \int \mathcal{D}A \exp\left\{\left\{\int d^Dx \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right)\right\}\right\}, \tag{A.1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and calculate the explicit form of $\langle W(C)\rangle$ in terms of the photon propagator

$$\Delta^{\mu\nu}(x-y) = g^{\mu\nu} \int \frac{d^Dp}{(2\pi)^D} \frac{e^{ip_\mu(x^\mu - y^\mu)}}{p^2} \tag{A.2}$$

in D dimensions. The first thing to do is to take the Fourier transform of the integral in d^Dx in (A.1). Doing it we get

$$\int d^Dx \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right) = \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\left[-A_\mu(k)M^{\mu\nu}A_\nu(-k) + J^\mu(k)A_\mu(-k) + J^\mu(-k)A_\mu(k)\right],$$

with $M^{\mu\nu} = k^2 g^{\mu\nu} - k^\mu k^\nu$ [22]. To solve the functional integration we may want to put this expression in a quadratic form in order to identify a Gaussian. To do such trick we would like to use the inverse of the matrix $M^{\mu\nu}$, but it does not have one, because it has zero determinant. To avoid this problem we will pick the Lorenz gauge $\partial_\mu A^\mu = 0$, that in the Fourier space reads $k^\mu A_\mu = 0$. This selects only the fields A^μ that satisfies the gauge, and we make the integration only over these fields, so the only contribution of $M^{\mu\nu}$ in the calculation is when $M^{\mu\nu} = k^2 g^{\mu\nu}$, which has the inverse $(M^{-1})^{\mu\nu} = \frac{g^{\mu\nu}}{k^2}$.

Now that we have solved the problem we can make the shift on the field to complete the square. Using

$$A_\mu(k) = A'_\mu(k) + J_\nu(k)(M^{-1})^{\mu\nu}, \tag{A.3}$$

and substituting it on the previous expression we get

$$\langle W(C)\rangle = \int \mathcal{D}A' \exp\left[\frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\left(J_\mu(k)(M^{-1})^{\mu\nu}J_\nu(-k) - A'_\mu(k)(M^{-1})^{\mu\nu}A'_\nu(-k)\right)\right],$$

that is exactly what we wanted to find. The $J_\mu(k)$ dependent term is only a constant to the integral, while the $A_\mu(k)$ term is a Gaussian integral. So, excluding constants of normalization, we get

$$\langle W(C)\rangle = \exp\left[\frac{1}{2}\int \frac{d^Dk}{(2\pi)} J_\mu(k)\frac{g^{\mu\nu}}{k^2}J_\nu(-k)\right],$$

and going back to coordinate space we get

$$\langle W(C)\rangle = \exp\left[\frac{1}{2}\int d^Dx d^Dy J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right], \tag{A.4}$$

with

$$\Delta^{\mu\nu}(x-y) = g^{\mu\nu}\int \frac{d^Dp}{(2\pi)^D}\frac{e^{ip_\mu(x^\mu-y^\mu)}}{p^2}.$$

The next step is to put the last expression in terms of the loop integrals, we can do it using the explicit form of the source given by

$$J^\mu(y) = q_e \oint_C \frac{dx^\mu}{dt_y}\delta^{(D-1)}(x-y).$$

Putting it on the expression (A.4) we get

$$\langle W(C)\rangle = \exp\left[\frac{q_e^2}{2}\int d^Dx d^Dy \oint_C \frac{dx'_\mu}{dt_x}\delta^{(D-1)}(x'-x)\Delta^{\mu\nu}(x-y)\oint_C \frac{dy'^\mu}{dt_y}\delta^{(D-1)}(y'-y)\right],$$

taking the time coordinates to be $t_x = x^D$ and $t_y = y^D$ we can cancel them, giving

$$\langle W(C)\rangle = \exp\left[\frac{q_e^2}{2}\oint_C dx'_\mu \oint_C dy'^\mu \int d^{(D-1)}x d^{(D-1)}y \delta^{(D-1)}(x'-x)\Delta^{\mu\nu}(x-y)\delta^{(D-1)}(y'-y)\right],$$

contracting the deltas we finally get

$$\langle W(C)\rangle = \exp\left[\frac{q_e^2}{2}\oint_C dx_\mu \oint_C dy^\mu \Delta^{\mu\nu}(x-y)\right], \tag{A.5}$$

exactly the expression (2.35).

B HOMOTOPY GROUP

Let us have two continuous mappings $f_1(x) : \mathcal{M} \to \mathcal{N}$ and $f_2(x) : \mathcal{M} \to \mathcal{N}$ from a manifold $\mathcal{M}$ to another $\mathcal{N}$, if we can construct a map $f(x,\tau)$ with $0 \leq \tau \leq 1$ such that $f(x,0) = f_1(x)$ and $f(x,1) = f_2(x)$ we say that $f_1(x)$ and $f_2(x)$ are homotopic. In other words it says that if we can continuously deform a map into another, they are said to be homotopic to each other[21, 16].

Giving many mappings we can construct an equivalence class between the ones that are homotopic to each other, making a set of inequivalent elements. If we consider closed paths called loops, we can identify them with an operation and construct a group, this is what happens when we are dealing with $k-$spheres S^k as manifolds. The set of homotopy classes of loops going from the $k-$sphere S^k to the manifold $\mathcal{N}$ is denoted by $\Pi_k(\mathcal{N})$. Let us take as example the case of $\Pi_1(\mathcal{N})$ with $\mathcal{N} = \mathbb{R}^2 - \{0\}$. It is clear that we can construct a closed path C_0 starting from x that can be contracted if it does not circle the origin. We can also define another loop C_0' that start from point x and do not circle the origin too, so we see that $C_0 \sim C_0'$. Now we can work with a path C_1 starting from x that circle the origin and then goes through C_0, the resultant path is described by $C_1' = C_1 + C_0$. Now, because we have the origin enclosed by the path, it is clear that C_1 is not equivalent to C_0, and since we can deform C_1' to C_1 we say that $C_1 + C_0 \sim C_1$. So we identify the following structure $C_0 + C_0' \sim C_0$, $C_0 + C_1 \sim C_1$, $C_1 + C_1' \sim C_2$ and so forth. Analysing this structure we can identify the closed loop C_0 with a null element and define a curve $-C$ that is the same as C but in the opposite direction, the curves that goes around the origin n times can be identified with C_n and the union of two loops is an operation. All this form a group of the integers $\mathbb{Z}$ with the addition as operation, this is what we call the homotopy group, for the case $\Pi_1(S^1)$ we have

$$\Pi_1(S^1) = \mathbb{Z}. \tag{B.1}$$

In the case of $n-$spheres we have that $\Pi_n(S^n) = \mathbb{Z}$, what is intuitive because if the function keeps constant in the point x we identify it with the null element, and any integer number of loops around the manifold can be associated with an integer. Another example is $\Pi_k(S^n) = 0$ for $k \leq n-1$, that can be understood taking the example of $\Pi_1(S^2)$, if we put a closed loop on the 2-sphere, any loop can be contracted to a point, leading to only the null element. The homotopy groups for compact $SU(n)$ groups can be summarized in [16]

$$\Pi_{2r+1}(SU(n)) = \mathbb{Z}, n-1 \geq r. \tag{B.2}$$

Now consider the parametrization for the mapping $g^a(x) : S^n_M \to S^n_N$ with $a = 1, 2, ..., n+1$ and $g^a g^a = 1$, we may want to construct an object that counts the number of times the manifold S^n_M covers S^n_N. Given the homotopy group $\Pi_n(S^n) = \mathbb{Z}$ we already know that the cover will be an integer number of times, but how does it manifest in terms of the mappings? Consider the volume of the $n-$sphere S^n_N given by

$$dV = \frac{1}{n!}\epsilon_{a_1 a_2 \ldots a_{n+1}} g^{a_1} dg^{a_2} \wedge dg^{a_3} \wedge \ldots \wedge dg^{a_{n+1}}, \tag{B.3}$$

if we use the parametrization $g^a(x)$ we can integrate this volume in the manifold S^n_M and divide it by the volume of the sphere, with it we define

$$Q[g] = \frac{1}{n! vol(S^n)} \int_{S^n_M} \epsilon_{a_1 a_2 \ldots a_{n+1}} g^{a_1} dg^{a_2} \wedge dg^{a_3} \wedge \ldots \wedge dg^{a_{n+1}}, \quad vol(S^n) = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}. \tag{B.4}$$

This is the winding number and it measures how many times the manifold S^n_M covers the volume of S^n_N, since we have this the homotopy group are the integers, then $Q[g] \in \mathbb{Z}$, and for every integer we have a whole equivalence class of mappings.

Let us now calculate the variation of the winding number under a continuous transformation in the map $g^a(x) \to g^a(x) + \delta g^a(x)$. Substituting it on (B.4) we get

$$Q[g + \delta g] - Q[g] = \frac{1}{n! vol(S^n)} \int_{S^n_M} \epsilon_{a_1 a_2 \ldots a_{n+1}} \delta g^{a_1} dg^{a_2} \wedge dg^{a_3} \wedge \ldots \wedge dg^{a_{n+1}},$$

guided by the relation $g^a g^a = 1$ we have that $g^a \delta g^a = 0$, and using the fact that the differential $\epsilon_{a_1 a_2 \ldots a_{n+1}} dg^{a_2} \wedge dg^{a_3} \wedge \ldots \wedge dg^{a_{n+1}}$ is proportional to g^{a_1} we have that $Q[g+\delta g] = Q[g]$, proving that the winding number is invariant under continuous transformation of the maps, making it a homotopic invariant.

C WARD IDENTITIES

In classical field theory we find the conserved currents of global symmetries using the Noether theorem

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi, \tag{C.1}$$

but nothing guarantees that this current will be conserved in the quantum theory too, this is why we need a new conservation law, the Ward identities.

Let us start with the field variation under a local symmetry

$$\phi' = \phi + \epsilon(x)\delta\phi, \tag{C.2}$$

the action variation will be

$$\begin{aligned}
\delta S &= \int d^D x \left(\epsilon(x)\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \epsilon(x)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) \right) \\
&= \int d^D x \left(\epsilon(x)\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\epsilon(x) \right] \delta\phi \right) \\
&= \int d^D x J^\mu \partial_\mu \epsilon(x), \quad \text{with} \quad J^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi,
\end{aligned} \tag{C.3}$$

when the symmetry is global this is identically null. With this result in hand we can go further in the demonstration of the Ward identities. The classical requirement for some symmetry is to keep the action invariant, but we saw that for local transformations it is not possible, and to derive quantum relations we need to make it local, so here we will use the symmetry condition for quantum mechanics, that is, we need the expected value of some operator to be invariant under the transformation. In the quantum field theory this is manifested using functional integration, so for the operator of fields $X = \prod_i \phi(x_i)$ we would like to have $\langle X \rangle = \langle X' \rangle$, with $X' = \prod_i \phi'(x_i)$, where we are using

$$\langle X \rangle = \frac{\int \mathcal{D}\phi X e^{iS[\phi]}}{\int \mathcal{D}\phi e^{iS[\phi]}}. \tag{C.4}$$

To verify the symmetry we will calculate

$$\int \mathcal{D}\phi X e^{iS[\phi]} = \int \mathcal{D}\phi' X' e^{iS'[\phi']}$$

and see what condition we can extract from it.

We will start by explicitly calculating $\langle X' \rangle$ with the variation of the fields (C.2)

$$\langle X' \rangle = \int \mathcal{D}\phi' \prod_i^n \phi(x_i) e^{iS'[\phi']} = \int \mathcal{D}\phi' [\phi(x_1) + \epsilon(x_1)\delta\phi(x_1)] \cdots [\phi(x_n) + \epsilon(x_n)\delta\phi(x_n)] e^{i(S+\delta S)},$$

considering that the integration measure $\mathcal{D}\phi'$ has a non-trivial Jacobian J, and taking only terms until first order in $\epsilon(x)$ we get

$$\langle X' \rangle = \int \mathcal{D}\phi J \left[X + \sum_j \epsilon(x_j)\phi(x_1)\cdots\delta\phi(x_j)\cdots\phi(x_n) \right] e^{iS}(1+\delta S) = \langle X \rangle \,.$$

Now we can expand our Jacobian in $J = 1 + i\int d^D x\epsilon(x)\mathcal{O}(x)$ and use the variation of the action given in (C.3), canceling the $\langle X \rangle$ terms we find

$$\int \mathcal{D}\phi e^{iS} \left(i\int d^D x\epsilon(x)\mathcal{O}(x)X + iX\int d^D xJ^\mu \partial_\mu \epsilon(x) + \sum_j \epsilon(x_j)\phi(x_1)\cdots\delta\phi(x_j)\cdots\phi(x_n) \right) = 0,$$

that in terms of the mean values is

$$i\int d^D x\epsilon(x) \langle \mathcal{O}(x)X \rangle - i\int d^D x\epsilon(x) \langle \partial_\mu J^\mu X \rangle = -\int d^D x\epsilon(x) \sum_j \delta(x-x_j) \langle \phi(x_1)\cdots\delta\phi(x_j)\cdots\phi(x_n) \rangle$$

which can be reduced to the Ward identities

$$\langle \partial_\mu J^\mu X \rangle = \langle \mathcal{O}(x)X \rangle - i\sum_j \delta(x-x_j) \langle \phi(x_1)\cdots\delta\phi(x_j)\cdots\phi(x_n) \rangle \,. \tag{C.5}$$

It is worth to discuss some features of this expression. The second term on the right side is equivalent to the one we find in the conservation rule in the presence of sources for the classical theory, it is called contact term. The first term though is not so simple, comparing it with the classical conservation and a non trivial Jacobian, the conservation law would be broken by it, this is called an anomaly, when a symmetry of the classical theory is not implemented in the quantum version.

D WILSON LOOP RELATION WITH THE POTENTIAL

Let us derive the expression (2.30). Considering the path in the Figure 6 we have the following Wilson loop VEV

$$\langle W(\Gamma)\rangle = \left\langle \exp i \left(\int_0^R A_i(0,x_1,x_2,x_3)dx^i + \int_0^T A_0(x^0,\vec{R})dx^0 \right.\right.$$
$$\left.\left. - \int_0^R A_i(T,x_1,x_2,x_3)dx^i - \int_0^T A_0(x^0,0,0,0)dx^0 \right) \right\rangle .$$

Using the gauge freedom we can set $A^0 = 0$, so we get

$$\langle W(\Gamma)\rangle = \langle \exp i \left(\int_0^R A_i(0,x_1,x_2,x_3)dx^i - \int_0^R A_i(T,x_1,x_2,x_3)dx^i \right) \rangle . \tag{D.1}$$

Defining the object

$$\psi(T) = \exp\left(-i \int_0^R A_i(T,x_1,x_2,x_3)dx^i \right) , \tag{D.2}$$

we have

$$\langle W(\Gamma)\rangle = \langle \psi(T)\psi^\dagger(0)\rangle . \tag{D.3}$$

Giving that, we can use a set of energy eigenstates and time translation to find

$$\begin{aligned} \langle W(\Gamma)\rangle &= \sum_n \langle 0| e^{HT}\psi(0)e^{-HT} |n\rangle \langle n| \psi^\dagger(0) |0\rangle \\ &= \sum_n e^{-E_n T} | \langle \psi\psi^\dagger \rangle |^2. \end{aligned} \tag{D.4}$$

When the limit of $T \to \infty$ is taken, we get

$$\langle W(\Gamma)\rangle \sim e^{-E_0 T}. \tag{D.5}$$

Where we consider the ground state energy equivalent to the potential energy.

E DUAL ACTION FOR D DIMENSIONS

The dual action can be made with the help of the dual field strength $\bar{F} = d\bar{A}$, where $\bar{A}$ is a $(D-3)$ - form. Its components can be given by

$$\frac{(-1)^{D-1}}{(D-2)}\bar{F}_{\mu_1\cdots\mu_{D-2}} = \partial_{[\mu_1}\bar{A}_{\mu_2\cdots\mu_{D-2}]}. \tag{E.1}$$

Starting with the Maxwell action for D dimensions

$$S = \int d^D x - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}, \tag{E.2}$$

we can insert the Bianchi identity with a Lagrange multiplier $\bar{A}$

$$\begin{aligned} S &= \int d^D x \left[-\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\bar{A}_{\mu_1\cdots\mu_{D-3}}\epsilon^{\mu_1\cdots\mu_D}\partial_{\mu_{D-2}}F_{\mu_{D-1}\mu_D} \right] \\ &= \int d^D x \left[-\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(\partial_{\mu_{D-2}}\bar{A}_{\mu_1\cdots\mu_{D-3}}\right)\epsilon^{\mu_1\cdots\mu_D}F_{\mu_{D-1}\mu_D} \right]. \end{aligned}$$

Substituting (E.1) on the expression above we find

$$S = \int d^D x \left[-\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(\frac{(-1)^{D-1}}{(D-2)}\bar{F}_{\mu_1\cdots\mu_{D-2}}\right)\epsilon^{\mu_1\cdots\mu_D}F_{\mu_{D-1}\mu_D} \right]. \tag{E.3}$$

Now integrating out $F^{\mu\nu}$ leads us to the effective dual action

$$S_{eff} = \int d^D x - \frac{e^2(-1)^D}{2(D-2)}\bar{F}^2. \tag{E.4}$$