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**SCALAR FIELDS, PROPAGATORS, AND
CONSEQUENCES OF EXPANDING UNIVERSES**

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Thesis presented to the Master's Program of Universidade Estadual de Londrina in partial fulfillment of the requirements for the degree of Master in Physics.

Supervisor: Prof. Dr. Sandro Dias Pinto Vitenti.

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*An infinite amount of time is required to be at peace
with the finitude of everything.*

ABSTRACT

KLUG, M.P.P. **Scalar fields, propagators, and consequences of expanding universes.** 2025. 60 f. Dissertação (Mestrado em Física) – Universidade Estadual de Londrina, Londrina, 2025.

Quantum Field Theory (QFT), being widely regarded as successful when it comes to describing particles, presents an important framework in the current landscape of modern physics to not only fundamental scales, but also large-scale phenomena, alongside General Relativity. The merge of both theories allow the investigation of particle behavior taking the universe expansion into account; this is done through the generalization of the field description to curved spacetimes. In this context, important quantities to usual QFT calculations— such as the Feynman propagator or vacuum states—are not easily or uniquely defined as they are in Minkowski spacetimes; thus, a close inspection of the modifications geometry brings to the field description is set. This thesis studies a free massive real scalar field in expanding universes that present spatially flat Friedmann- Lemaître-Robertson-Walker metrics, aiming to find the modes that describe the fields over time by solving the appropriate equation of motion in order to understand the field's evolution. Exact solutions are found by virtue of the Frobenius method, providing closed modes to compose the field. With them, the phenomenon of particle creation between different frames is discussed and, alongside the behavior of the modes themselves, evaluated.

Palavras-chave: FLRW; Klein-Gordon; QFT; Curved Spacetimes; Scalar Field.

RESUMO

KLUG, M.P.P. **Campos escalares, propagadores, e consequências de universos em expansão.** 2025. 60 p. Master's Thesis (Master in Science in Physics) – State University of Londrina, Londrina, 2025.

Teoria Quântica de Campos (TQC), sendo amplamente considerada como um sucesso quando se trata da descrição de partículas, apresenta uma estrutura importante no cenário da Física atual, não apenas para escalas fundamentais, mas também quando se trata de fenômenos de grandes escalas, em conjunto com a Relatividade Geral. A união de ambas as teorias permite a investigação do comportamento de partículas ao levar a expansão do universo em consideração; isto é feito através da generalização da descrição de campos para espaços-tempos curvos. Neste contexto, quantidades importantes para cálculos de TQC padrão—como propagadores de Feynman ou estados de vácuo—não são fáceis ou unicamente definidas, como são em espaços-tempos de Minkowski; desta forma, uma inspeção atenta sobre as modificações trazidas pela geometria à descrição de campos é estabelecida. Esta dissertação estuda um campo escalar real massivo livre em universos em expansão que apresentam métricas de Friedmann-Lemaître-Robertson-Walker espacialmente planas, aspirando encontrar modos que descrevem os campos ao longo do tempo ao solucionar a equação de movimento apropriada, a fim de entender a evolução do campo. Soluções exatas são encontradas por meio do método de Frobenius, provendo modos fechados que compõem o campo. Com eles, o fenômeno de criação de partículas entre diferentes referenciais é discutido e, juntamente com o comportamento dos modos em si, avaliados.

Key-words: FLRW; Klein-Gordon; TQC; Espaços Curvos; Campo Escalar.

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1 INTRODUCTION

With the advent of Quantum Field Theory (QFT) in the last century [1–6], physicists began to value the fundamental description the field framework brought to the understanding of nature. With that running in parallel with the theoretical development of General Relativity (GR) [7–14] and the observations which would reveal the structures of the universe in its largest scales that would soon follow [15, 16], it did not take long for the field interpretation and formalism to be applied in these cosmological scales as well [17–21]. One of the first approaches would be that of scalar fields in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe—two of the simplest elements of their respective areas of study. Scalar field models are able to describe phenomena such as inflation [22–26], early-universe structure formation [27, 28], dark matter [29–33], and—as recent observations suggest—dark energy [34–37]. Despite the simplicity of both the field and the metric, the changes a dynamical geometry brings to the field framework, absent in its standard form dealt in Minkowski space, are not to be underestimated, as many phenomena emerge from the merge of the two theories [20, 38–41]; given the vast range of application they serve in cosmological studies, however, the arising complications are worth the investigation.

In order to understand the behavior of scalar fields it is necessary to analyze their evolution, encoded in flat spacetimes by the Klein-Gordon equation; their functional form, however, is affected by the geometry, so when studied in an expanding universe, the appropriate equation of motion must be considered. The Klein-Gordon equation in FLRW metrics (the FLRW-KG equation) is well known, yet exact solutions to them is an understudied subject in the literature; this is somewhat surprising given the approachability of these types of models, compared to higher spin fields and more complicated geometries, added to their usefulness in Cosmology. Closed solutions have been found, though only for specific instances [42–45], regarding backgrounds that present ideal symmetries; and even in those limited scenarios the solutions are given in terms of special functions; on top of that, these instances often consider massless fields that—although simpler to deal with—do not carry the same generality as arbitrarily massive fields, hindering their ability to describe common physical occurrences, leaving models with general field mass m and wavenumber k in various universes containing different fluids yet to be solved.

This unexplored space in the literature motivates the inspection of the FLRW-KG equation with the intent to find scalar modes, allowing observables such as expected number of particles to be forecast for a plethora of universes with different backgrounds and fields with general m and k ; as well as the quantity that informs how a particle propagates throughout spacetime: the propagator. Moreover, the aim to find modes in exact form is driven by their computational efficacy over numerical evaluations required by integral form that special functions present [46, 47].

In order to solve the FLRW-KG equation, it is first needed to find its explicit form. This is done by modeling the expansion of the universe by FLRW metrics with scale factors that respect a time power law $a(\eta) = \eta^\lambda$; this parametrization is reasonably simple to be dealt with and follows naturally from GR for a barotropic fluid with constant equation of state [48–51]. The appropriate equation of motion is then solved using the Frobenius and the reduction of order methods. With the solutions in hand, particle creation can be calculated using Bogolyubov transformations [41, 45, 52–58] and the propagator can be explicated for a very general class of scalar fields in FLRW backgrounds.

This thesis is organized as follows: In Chapter 2 the base for working with QFT is laid out, alongside with its generalization to curved spacetimes, discussing the evolution of scalar fields and the construction of the propagator associated with their particles; in Chapter 3 the geometry and the modeling of the universes at matter are set, and the techniques to solve the FLRW-KG equation are applied; Chapter 4 analyses physical consequences of the universe expansion and the behavior of the found modes, culminating in the explicit form for the scalar propagator in FLRW universes; lastly, Chapter 5 provides final remarks over what has been studied. Additionally, Appendix A briefly introduces the volume differential form, and Appendix B generalizes the Frobenius solution to general coupling.

2 QUANTUM FIELD THEORY

Motivated to describe the contents of the universe and their interactions with themselves, one can resort to a field description. Under it, what is called a particle is no longer a point-like object as in the classical view, but instead, a field excitation; and so is any possible combination of particles amounting to any body out there in space. The Earth itself and everything living on its surface can—in theory—be described through these lenses, as well as everything existing in its core, often modeled by subatomic particles under quantum chromodynamics. Not only terrestrial objects benefit from this framework, but also less understood ones—dark matter, for instance, is often modeled as a scalar field [29–33]. With its potential to explain phenomena from the most fundamental scales, through effective descriptions, to the greatest cosmological ones, QFT stands as a vital framework to current physics, and invites exploration of its tools in order to bring clarification on models not yet completely understood.

2.1 Standard QFT

This thesis aims to make a field description of scalar particles on an expanding geometry. But before engaging in such a matter, it is prudent to introduce the formalism and features of QFT in a much simpler context. The standard approach in flat Minkowski space is a good starting place.

2.1.1 The Scalar Field

Much like the procedure from classical to quantum mechanics, where variables, such as position q and momentum p , are promoted to operators through canonical commutation relation (assuming the natural units system, where $\hbar = 1$),

$$[q_i, p_j] = i\delta_{ij}, \quad (2.1)$$

$$[q_i, q_j] = [p_i, p_j] = 0, \quad (2.2)$$

QFT does the same, but instead of treating position and momentum of a given object as ordinary variables, it treats them as continuous fields, and imposes new relations with a (three-dimensional) Dirac delta, instead of a Kronecker one:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}), \quad (2.3)$$

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0, \quad (2.4)$$

with the scalar field ϕ playing the analogous role of position, and π its associated momentum coming from the canonical Lagrangian—shortly to be shown:

$$\pi(\vec{x}, t) = \frac{\partial}{\partial t}\phi(\vec{x}, t). \quad (2.5)$$

These fields are to be thought of as permeating the entire spacetime, such they depend explicitly on the spatial part of the four-dimensional vectors $x = (t, \vec{x})$ and $y = (\vec{t}, \vec{y})$, having associated values for each event. Through Fourier transform, they can be generally decomposed into an imaginary exponential part, denoting its oscillating properties, and a creation ($a_{\vec{p}}$) and annihilation ($a_{\vec{p}}^\dagger$) operators part:

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right), \quad (2.6)$$

$$\pi(\vec{x}) = -i \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right). \quad (2.7)$$

Here, $\omega_{\vec{p}}$ is the oscillation frequency of an excitation associated with the momentum \vec{p} , generated by the creation operator $a_{\vec{p}}^\dagger$ over a nonexcited state $|0\rangle$:

$$a_{\vec{p}}^\dagger |0\rangle = |\vec{p}\rangle. \quad (2.8)$$

Conversely the annihilation operator is the one defined to annihilate such state:

$$a_{\vec{p}} |0\rangle = 0. \quad (2.9)$$

From the commutation relations (2.3) and the field expressions (2.6) and (2.7), it follows that the ladder operators¹ also respect one:

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^{\frac{3}{2}} \delta^{(3)}(\vec{p} - \vec{q}). \quad (2.10)$$

It is important to note that although the fields do depend on the ladder operators, whose presence brings a discrete notion upon the accessible states, said operators are being integrated over all possible momenta, spanning over all possible combinations of field configurations. Furthermore, if one is to behold (2.8) and interpret the state $|\vec{p}\rangle$ as a single particle brought into existence by the creator operator—as one is led to by analyzing the energy spectrum of the field—by integrating over all momenta, one would be able to combine particles to create more complex structure; at first two particles with the same momentum; but soon, a mole of them, with a rich variety of (underlyingly) discrete momenta, possibly interacting with even particles from a different field, eventually compounding themselves into complex monkey-sized bodies or cosmological formations.

A free real scalar field have the Lagrangian density (often referred to simply as the Lagrangian)

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2), \quad (2.11)$$

with m being the mass of the field, which under the field theory point of view carries the information of resistance to change in values of ϕ . Integrating over spacetime the action presents itself as

$$S = \int d^4x \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2), \quad (2.12)$$

which, after extremized, provides the equation of motion

$$(\partial_\mu \partial^\mu + m^2) \phi(x) = 0. \quad (2.13)$$

This is the *Klein-Gordon equation*; it dictates the dynamics of a scalar field.

To build the action, the Lagrangian had to be integrated over the four dimensions the derivatives ∂_μ refer to, with μ running from 0 to 3. However, in order to associate them with spacetime, as it was mentioned to be the case, it is necessary to assume the natural units system, where the speed of light is equal to the

¹ Not to be confused with the angular momentum operator from standard quantum mechanics $J_\pm = J_x \pm iJ_y$. A ladder operator is an operator that increases or decreases the eigenvalue of another state. In the case of harmonic oscillators—which is essentially what QFT is about—the creator and annihilator play precisely that role.

reduced Planck constant, and both of them are equal to 1; $c = \hbar = 1$; only then the coordinates can be associated with space and time and the equations carry physical meaning. It also seems convenient to bring up the notation used for the metric signature in this work; namely the mostly minus, $(+, -, -, -)$. This is due to the vast majority of references regarding QFT choosing to adopt this one. Since the formalism had to be first studied from the ground up in order to achieve application in the cosmology field—even though the references of the latter often prefer the mostly plus signature—the convention was chosen to be consistent with QFT textbooks. Despite the common, and often heated, preferences for one in favor of the other, the actual development and results of the theories here discussed are independent of human opinions.

Turning to equation (2.13), it tells that a function $\phi(x)$ differentiated twice has to be equal to itself times $-m^2$, which is generally respected by the form

$$\phi_{\vec{k}}(x) = Ae^{-ik_\mu x^\mu} + Be^{+ik_\mu x^\mu}, \quad (2.14)$$

with A and B being constant coefficients, if k respects

$$k_\mu k^\mu = m^2. \quad (2.15)$$

The last equation is much too familiar, reminiscent of the relativistic dispersion relation; and indeed if k is to be considered the four-momentum, the function $\phi_{\vec{k}}$ in the form (2.14) corresponds to a plane-wave solution with wave-vector $k = (\omega, \vec{k})$. This alludes to the wave-like nature of the scalar field QFT brings to the table, and is in accordance with (2.6); only there the quantities are three-vectors instead of four-vector due to the decomposition being in the Schrödinger picture, while being analyzed in a given instant of time. If one were to write the fields in the Heisenberg picture, they would read

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip_\mu x^\mu} + a_{\vec{p}}^\dagger e^{+ip_\mu x^\mu} \right) \Big|_{p^0=E_{\vec{p}}}, \quad (2.16)$$

$$\pi(\vec{x}, t) = \frac{\partial}{\partial t} \phi(\vec{x}, t), \quad (2.17)$$

with $E_{\vec{p}} = \omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$ being the energy of the particle associated with the field. The distinction between p and k also becomes trivial, due to the use of the natural units system; $p = \hbar \left(\frac{\omega}{c}, \vec{k} \right) = (\omega, \vec{k}) = k$.

The present oscillatory behavior of the fields is what builds the foundation of their propagation throughout spacetime, which eventually leads to interactions, given an appropriate Lagrangian describing the system. As far as the free theory is concerned, however, one needs only to worry about tree-level transitions—the lowest-order in perturbative calculations—which are much more accessible. Still, one element is yet missing to properly propagate the field excitations.

2.1.2 The Green's Function

For what may seem a digression, one could pose the question "What happens when an operator acts on a function in such a way that the result is always a Dirac delta function?". Namely, for a linear operator L , impose that the function G always returns

$$LG(u, s) = \delta(u - s), \quad (2.18)$$

for all u and s . Of course, if the above relation is to be respected, the acting of L over G has to be zero everywhere in the function's domain, except when $u = s$, where it diverges; this sheds a notion of inverse onto

the function, in regard to the operator. This equation filters special values for the variables—or rather special combinations of values—and can be useful when trying to study relations between operators and functions that are for the most part trivial, but under the filtered circumstances carries notable meaning. The former question, although arguably unnatural, were in the minds of great mathematicians and physicists from the early 19th century, who were well aware of its importance, specially in the context of electrodynamics, and extensively studied equation (2.18) in the decades to come. Thus the function G became known as the Green’s function of the linear operator L , named after the British mathematical physicist George Green.

Beyond the 19th century, Green’s functions continued being studied; not only because of the scrutiny of works on classical electromagnetism, as Green himself laid the groundwork for [59], but also in the quantum domain, as the formalism for QFT began to be developed [60–62]. They proved to be a class of functions that, owing to their prominent particularities, often point out to quantities of interest, that one might want to watch over.

In the context of QFT—as it shall be seen next—the filtering brought by Green’s functions can be used to relate points in spacetime; as such, excitations given in different events can be compared through a quantity built from the structure here discussed. In other words, the Green’s functions provide a way to construct an object that propagates states through spacetime.

2.1.3 The Klein-Gordon Propagator

When dealing with particles—field excitations—the field $\phi(x)$, which is composed of ladder operators with a given spectrum, operates over kets to create or annihilate states. This renders the state $\phi(y)|\alpha\rangle$ different from one $\phi(x)|\beta\rangle$, for arbitrary $|\alpha\rangle$ and $|\beta\rangle$ and events x and y . In order to compare the states’ overlap, one should be projected into the other; resulting in the probability amplitude of $|\alpha\rangle$ to propagate to $|\beta\rangle$:

$$D_{\alpha-\beta}(x, y) = \langle\beta|\phi(x)\phi(y)|\alpha\rangle. \quad (2.19)$$

If, for simplicity, the states before the acting of the field operators are to be considered the same—for instance, the vacuum of the same set of ladder operators—the amplitude then relates to the probability that a particle propagates from a point in spacetime to another. By expanding the fields through equation (2.16), only one of the four terms survives; namely, $\langle 0|a_{\vec{p}}a_{\vec{q}}^\dagger|0\rangle$. The amplitude then reads

$$D(x, y) = \langle 0|\phi(x)\phi(y)|0\rangle \quad (2.20)$$

$$= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{2E_{\vec{p}}} e^{-ip_\mu(x^\mu - y^\mu)}. \quad (2.21)$$

Although the integral form above is crucial while doing practical calculations in QFT (particularly when calculating Feynman diagrams and differential cross sections), it is of little importance in this thesis, that focuses on the intricacies of the field behavior and the propagator itself; matter that comes well before such calculations. The attention must then be put into the former form.

Keeping in mind that x and y are events in spacetime, with three spatial and one time component each, the probability amplitude can be interpreted in three different scenarios:

- For $x^0 > y^0$, $D(x, y)$ tells how likely it is to find a particle at \vec{x} at a later time x^0 if it was initially at \vec{y} at initial time y^0 .
- For $x^0 = y^0$, $D(x, y)$ tells how likely it is to find a particle at \vec{x} at time x^0 , if it was initially at \vec{y} at the same instant of time.

- For $x^0 < y^0$, $D(x, y)$ tells how likely it is that a particle was at \vec{x} at an earlier time x^0 if it is found at \vec{y} at a later time y^0 .

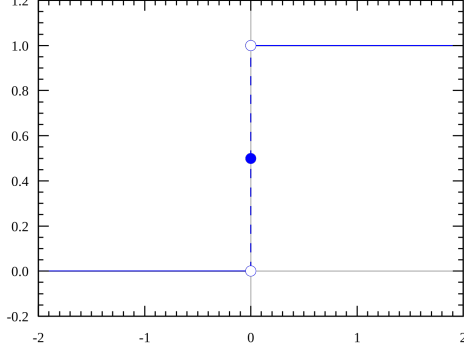


Figure 1 – Heaviside step function value (in the vertical axis) in terms of x (in the horizontal axis).

To inform only about particles propagating to the future, solely the first case should be considered. This can be achieved by taking the commutator of the field operators and multiplying the amplitude D by the Heaviside step function $\theta(x^0 - y^0)$, which equals 0 for $x^0 < y^0$, and 1 for $x^0 > y^0$, as Figure 1 shows, yielding:

$$D_R(x, y) = \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle. \quad (2.22)$$

This new amplitude considers purely final states in the future, and informs about how a configuration of the field influences posterior ones. Succinctly, it propagates particles from the past to the future; hence it is called the *retarded propagator*.

Analogously, it is possible to construct a function that analyzes the likeliness of the conception of an anterior field configuration given a posterior one, by considering purely final² states in the past. By taking the commutator and multiplying the original probability amplitude by $\theta(y^0 - x^0)$ (with the sign of the argument flipped), the *advanced propagator* is built:

$$D_A(x, y) = \theta(y^0 - x^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle. \quad (2.23)$$

With both these propagators, propagation and interactions between scalar particles can be studied. They, however, are not the only ones that can be defined, and it is often useful when carrying out calculations in QFT, to have a propagator that can take into consideration both particles known to have been realized in the past as well as in a future instant. In order to cover both cases without having to invoke more than one function every time, a third one is presented; the *Feynman propagator*, defined the following way:

$$D_F(x, y) = \begin{cases} D(x, y), & \text{if } x^0 > y^0, \\ D(y, x), & \text{if } x^0 < y^0 \end{cases} \quad (2.24)$$

$$= \theta(x^0 - y^0) \langle 0 | \phi(x)\phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y)\phi(x) | 0 \rangle \quad (2.25)$$

$$= \langle 0 | T\phi(x)\phi(y) | 0 \rangle. \quad (2.26)$$

² Here, "final" is meant as the one being compared to a set "initial" one. Said final state is often in the left side of the operators, represented by bracs, regardless of the actual time it was conceived.

In the first form, it is being expressed by the original amplitude, with the first argument always being the later one, and the second the earlier one. In the second form the same is written but openly, with the later field always being in the left while the earlier is always on the right. The third form condenses the notation utilizing the *time-ordering operator* T , defined by

$$T\phi(x)\phi(y) \equiv \begin{cases} \phi(x)\phi(y) & \text{if } x^0 > y^0, \\ \phi(y)\phi(x) & \text{if } x^0 < y^0. \end{cases} \quad (2.27)$$

The time-ordering operator may be interpreted as ordering the operators it acts upon in such a way that, if the latter were to act over the initial state, they would do so in chronological order; the rightmost operator first, then the following one. It can, in fact, be generalized to encompass various fields, always acting them over the initial state from the oldest operator, to the most recent one.

It may be noted that the second case, when $x^0 = y^0$, has not been given attention when building the propagators; such is the case as these expressions are commonly found inside integrals, rendering a single point of the function's value negligible, mattering only the values the function assumes for continuous range of their argument.

Thus, the Feynman propagator (2.26) accounts for the propagation of particles throughout spacetime. Furthermore, it also bears the notable feature of being a Green's function of the Klein-Gordon operator. By looking at equation (2.13), the linear operator $(\partial_\mu\partial^\mu + m^2)$ is identifiable as the Klein-Gordon operator, which acting upon the Feynman propagator returns

$$\begin{aligned} & (\partial_\mu\partial^\mu + m^2) \left[\theta(x^0 - y^0) \langle 0 | \phi(x)\phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y)\phi(x) | 0 \rangle \right] = \\ & = -\delta(x^0 - y^0) \partial_0 \langle 0 | \phi(x)\phi(y) | 0 \rangle + \delta(x^0 - y^0) \left[\partial_0 \langle 0 | \phi(x)\phi(y) | 0 \rangle + \cancel{\partial_i \langle 0 | \phi(x)\phi(y) | 0 \rangle} \right] + \\ & + \delta(x^0 - y^0) \left[\partial_0 \langle 0 | \phi(x)\phi(y) | 0 \rangle - \cancel{\partial_i \langle 0 | \phi(x)\phi(y) | 0 \rangle} \right] + \\ & + \delta(y^0 - x^0) \partial_0 \langle 0 | \phi(y)\phi(x) | 0 \rangle - \delta(y^0 - x^0) \left[\partial_0 \langle 0 | \phi(y)\phi(x) | 0 \rangle + \cancel{\partial_i \langle 0 | \phi(y)\phi(x) | 0 \rangle} \right] + \\ & - \delta(y^0 - x^0) \left[\partial_0 \langle 0 | \phi(y)\phi(x) | 0 \rangle - \cancel{\partial_i \langle 0 | \phi(y)\phi(x) | 0 \rangle} \right] + \\ & + \theta(x^0 - y^0) (\partial_\mu\partial^\mu + m^2) \langle 0 | \phi(x)\phi(y) | 0 \rangle + \theta(y^0 - x^0) (\partial_\mu\partial^\mu + m^2) \langle 0 | \phi(y)\phi(x) | 0 \rangle \end{aligned} \quad (2.28)$$

$$\begin{aligned} & = -\delta(x^0 - y^0) \langle 0 | \pi(x)\phi(y) | 0 \rangle + 2\delta(x^0 - y^0) \langle 0 | \pi(x)\phi(y) | 0 \rangle + \\ & + \delta(y^0 - x^0) \langle 0 | \phi(y)\pi(x) | 0 \rangle - 2\delta(y^0 - x^0) \langle 0 | \phi(y)\pi(x) | 0 \rangle \end{aligned} \quad (2.29)$$

$$= -\delta(x^0 - y^0) \langle 0 | \left(\pi(x)\phi(y) - \phi(y)\pi(x) \right) | 0 \rangle + 2\delta(x^0 - y^0) \langle 0 | \left(\pi(x)\phi(y) - \phi(y)\pi(x) \right) | 0 \rangle \quad (2.30)$$

$$= -i\delta^4(x - y). \quad (2.31)$$

Here, it is important to remember that the partial derivatives are in respect to the x -coordinates, $\partial_\mu = \frac{\partial}{\partial x^\mu}$; as well as the Dirac delta's evenness and the property $\delta'f = -\delta f'$. In the last line of equation (2.28) two Klein-Gordon equations were identified, which are equal to 0. In equation (2.30), the canonical commutation relation between the scalar field and its associated momentum was identified, allowing to reach a final form which is directly proportional to a four-dimensional Dirac delta; identical, apart merely from the negative imaginary constant; a simple rescaling that could be absorbed by the operator that acts over the function. This perhaps confusing, but ultimately simple, calculation permits the recognition of Green's functions status upon the propagator.

Having built the object that allows the study of particles transitions between states, and acknowledged its relation to a class of unique functions, let the scope be broadened and see how this all unfolds as the universe does so.

2.2 QFT in Expanding Geometry

In large scales—cosmological ones—the universe can be considered, to a good margin, spatially homogeneous and isotropic [48, 63–65]. These spatial characteristics can be described, whilst also taking the universe expansion across time into consideration, with the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, whose line segment for flat space sheets (following the convention mentioned in Section 2.1.1) is

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2. \quad (2.32)$$

The metric of such space corresponds to a spatially flat geometry that expands (or contracts) over time with the scale factor $a(t)$, leaving the curvature present in it to be reflected purely temporally. A change of parameters can be performed in order to leave the metric in a conformally invariant form, which can be algebraically useful, and is achieved through the introduction of conformal time η , that relates to the usual cosmic time t by $d\eta = \frac{dt}{a(t)}$. The line segment is henceforth written as

$$ds^2 = a^2(\eta)(d\eta^2 - d\vec{x}^2). \quad (2.33)$$

The associated metric tensor $g_{..}$ is³, explicitly,

$$g_{..} = \begin{pmatrix} a^2(\eta) & 0 & 0 & 0 \\ 0 & -a^2(\eta) & 0 & 0 \\ 0 & 0 & -a^2(\eta) & 0 \\ 0 & 0 & 0 & -a^2(\eta) \end{pmatrix} \quad (2.34)$$

In a new curved geometry, it is only to be expected that the fields do not behave exactly the same way as they do in flat ones. The excitations, described by wave behavior, are distributed across space; as time goes on they are stretched or squished, rendering the plane-wave description only locally good, and only in certain regimes. To reflect these changes, a new rendition of their dynamics is required.

2.2.1 The Scalar Field in FLRW

With a curved spacetime, an ordinary derivative holds no longer the intended meaning, requiring the incorporation of the metric alongside it in order to account for the changes between different points of the space, as well as the use of the covariant derivative ∇ compatible with the metric, $\nabla_\mu g_{\mu\nu} = 0$, updating the notion of 4-volume to the new geometry. Under these lights, the action of a real scalar field, minimally coupled⁴, becomes

$$S = \frac{1}{2} \int dt d^3\vec{x} \sqrt{|g|} (g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - m^2 \phi^2). \quad (2.35)$$

It then follows that the equation of motion of such a system reads

$$\left[\frac{1}{\sqrt{|g|}} \partial_\mu \left(g^{\mu\nu} \sqrt{|g|} \partial_\nu \right) + m^2 \right] \phi = 0 \quad (2.36)$$

³ The dots in place of the subscripts are to remind the position of the indices when representing the metric elements, contrasting with the elements of the inverse metric $g^{..}$. Moreover, it is useful to differ from the metric determinant g .

⁴ The free scalar field action (2.35) is symmetric by the implementation of a term proportional to the Ricci scalar R ; its presence evokes the notion of a gauge choice, and some works believe it to carry physical meaning when looking at cosmological data and their compatibility to theoretical models [66, 67]. Although of not unanimous importance, it is possible to generalize the discussion in order to take them into consideration. Appendix B approaches the results of this work taking general coupling into account.

The above is the general Klein-Gordon equation; the equation that dictates the dynamics of a scalar field for arbitrary geometries.

In the conformal time picture, the action and the equation of motion in FLRW stand as

$$S = \frac{1}{2} \int d\eta d^3\vec{x} a^2(\eta) [\phi'^2 - (\nabla\phi)^2 - m^2 a^2(\eta)\phi^2], \quad (2.37)$$

$$\phi'' + 2\frac{a'(\eta)}{a(\eta)}\phi' - \nabla^2\phi + m^2 a^2(\eta)\phi = 0, \quad (2.38)$$

where ' denotes derivatives in respect to conformal time η . Once again, ϕ is being spatially described as a plain wave excitation, therefore the Laplacian in the third term of equation (2.38) can be identified as $-k^2\phi$ —this can be seen by differentiating the plane wave form (2.14) twice in respect to position. Doing so, exposes to view the *FLRW-KG equation*, which stands at the core of scalar fields behavior and particles description in an expanding universe:

$$\phi'' + 2\frac{a'(\eta)}{a(\eta)}\phi' + [k^2 + m^2 a^2(\eta)]\phi = 0. \quad (2.39)$$

It may be noted that despite the field here used, ϕ , respecting the relativistic dispersion relation, it does not have the universe expansion—the geometry—explicitly incorporated in itself; it relies on the FLRW-KG equation to take that dependence into account when having its dynamics described. It is, however, possible to make a description where an auxiliary field is introduced, incorporating the scalar factor through a canonical transformation [41, Chapter 6]. Although the latter choice of representation has interesting consequences, for now the focus shall be on the local field interpretation, leaving to the equations themselves to carry the physical meanings of an evolving universe.

Performing a functional derivative in respect to ϕ' on the Lagrangian, it is possible to determine the conjugate momentum associated with the field in this geometry through the action's (2.37) integrand.

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\phi'} = a^2(\eta)\phi'(x). \quad (2.40)$$

For the very reason the field is described locally, its commutation relations continue to be as they were in flat spacetime, in spite of the momentum no longer being a direct time derivative of ϕ :

$$[\phi(\eta, \vec{x}), \pi(\eta, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}), \quad (2.41)$$

$$[\phi(\eta, \vec{x}), \phi(\eta, \vec{y})] = [\pi(\eta, \vec{x}), \pi(\eta, \vec{y})] = 0. \quad (2.42)$$

However, it is crucial to point out that these relations are only valid when evaluated at the same time; when evaluated at different points in time, each field's set of annihilation and creation operators may interact with one another, rendering $[\phi(x), \phi(y)] \neq 0$.

2.2.2 The Appropriate Green's Function

The Green's function (GF) has already been mentioned to select a special class for a given operator by imposing the restriction that the function, under the operator's action, must return a divergent value in a particular way for a set of points, and zero everywhere else, according to a delta distribution, as equation (2.18) suggests. However, as a mathematician would certainly point out, said equation used to define what a GF is does not really make formal sense as an equation for functions, unless it is written inside an integral

and only in a distributional sense—much like any isolated Dirac delta function would not. Of course, most of the time these deltas appear they are innocuous to the actual calculations being performed, given they either can be neglected due to the quantities at interest being comparison of values, instead of absolute ones, or they respect the formerly mentioned case and are actually being integrated over a region in spacetime. Nevertheless, in the GF case, this nuance is not to be overlooked. If, while building a notion of GFs, one does not concern themselves with the integration process—posterior to the action of the operator—one would end up with a class of functions that does not necessarily represents what it was intended to; comparable to how an ordinary derivative no longer conveys the intended meaning once curvature is introduced. In a careful manner, the condition to be imposed should not be $LG(u, s) = \delta(u - s)$, but instead, the integration of it. Doing so, would account for the 4-volume measurement of the space, incorporating the geometry in the definition of Green's functions.

In order to integrate quantities throughout arbitrary geometries the appropriate *volume element* is necessary. It is properly defined by a differential form of the same degree of the space's dimension n [48, Chapter 2], as Appendix A details; for now, suffice to say that for a quantity I scattered across spacetime following a density distribution function $f(u)$, its value is computable through the integral

$$I = \int f(u) \sqrt{|g|} d^n u, \quad (2.43)$$

where the absolute value of the determinant of the metric $|g|$ helps construct the volume element $|g|d^n u$.

This means that the GF condition should be

$$\int \sqrt{|g|} d^n u LG(u, s) = \int d^n u \delta(u - s) = 1. \quad (2.44)$$

In FLRW the volume element is $a^4(\eta) d\eta d^3 \vec{x}$, and the condition is reduced to

$$a^4(\eta) LG(x, y) = \delta^4(x - y), \quad (2.45)$$

$$\Rightarrow LG(x, y) = \frac{1}{a^4(\eta)} \delta^4(x - y). \quad (2.46)$$

Equation (2.46) tells that in order for a function $G(x, y)$, whose arguments are part of a spacetime with FLRW metric, to bear the meaning that a GF bears in flat spaces, its definition must carry a measure factor that takes the geometry into account; this factor is the inverse of the determinant of the metric; essentially the volume element; and by requiring so, reframes quantities that filter special combination of values under the universe expansion.

2.2.3 The FLRW-KG Propagator

It was seen that the construction of the Green's function changes according to the geometry; a natural question that follows is if something similar happens to the propagator. The notion of projecting past and future states was used when defining it in Minkowski spacetime, and it is reasonable to reuse it when trying to generalize it to a more complex space. In fact, the flat Feynman propagator is a good starting place. It should be checked what happens when the FLRW-KG operator, present in (2.38), acts upon the Feynman propagator. First, the operator is explicitly identified. From the general form (2.36), the FLRW metric is

applied:

$$\begin{aligned}
\Gamma &\equiv \frac{1}{\sqrt{|g|}} \partial_\mu \left(g^{\mu\nu} \sqrt{|g|} \partial_\nu \right) + m^2 \\
&= \frac{1}{\sqrt{a^8}} \left[\partial_0 \left(\frac{1}{a^2} \sqrt{a^8} \partial_0 \right) + \partial_i \left(-\frac{1}{a^2} \sqrt{a^8} \partial_i \right) \right] + m^2 \\
&= \frac{1}{a^2} \left[\partial_0 \partial_0 + 2 \frac{a'}{a} \partial_0 + k^2 + m^2 a^2 \right],
\end{aligned} \tag{2.47}$$

where the spatial derivative can be turned into the matrix elements on a Fourier basis when considering spatially flat sections, so k^2 can be identified in the last line. Thus, the FLRW-KG linear operator Γ is defined. Once acting it upon a scalar field, one has the FLRW-KG equation (2.39).

From the equation above on out the index 0, as in ∂_0 or x^0 , will refer to the zeroth component in the conformal time framework—not to be confused with cosmic time, which shall not be used henceforward—and so is the dependence on η implicit in the scale factor a .

Aside from the FLRW-KG operator, a time-ordered projection between initial state $|\alpha\rangle$ and final state $|\beta\rangle$ is given by $\langle\beta|T\phi_\beta(x)\phi_\alpha(y)|\alpha\rangle$, where the fields subscripts relates to the creation and annihilation operators they can be decomposed into; namely ϕ_α can be decomposed in ladder operators ($a_{\vec{q}}, a_{\vec{q}}^\dagger$) associated with its vacuum, $a_{\vec{q}}|0_\alpha\rangle = 0$, and ϕ_β is decomposed into another set of operators ($b_{\vec{p}}, b_{\vec{p}}^\dagger$) associated with its own vacuum, $b_{\vec{p}}|0_\beta\rangle = 0$. The propagator informs about the probability amplitude of one state transitioning to another. One can say that it plays its role indirectly comparing states $|\alpha\rangle$ and $|\beta\rangle$ by sandwiching itself between them; or directly comparing states created by the acting of the creation and annihilation operators inside the fields, $\langle\beta|T\phi_\beta(x)\phi_\alpha(y)|\alpha\rangle \propto \langle\beta|b_{\vec{p}}a_{\vec{q}}^\dagger|\alpha\rangle$; in the latter case the states being directly compared are $a_{\vec{q}}^\dagger|\alpha\rangle$ (initial) and $b_{\vec{p}}^\dagger|\beta\rangle$ (final); either way the propagator projects initial and final states wished to be compared. The explicit description of the fields shall be further inspected in Chapter 4—for now, the attention is turned to the acting of the linear operator Γ onto the projection:

$$\begin{aligned}
\Gamma \langle\beta|T\phi_\beta(x)\phi_\alpha(y)|\alpha\rangle &= \frac{1}{a^2} \left[\partial_0 \partial_0 + 2 \frac{a'}{a} \partial_0 + k^2 + m^2 a^2 \right] \\
&\quad \cdot \left[\theta(x^0 - y^0) \langle\beta|\phi_\alpha(x)\phi_\beta(y)|\alpha\rangle + \theta(y^0 - x^0) \langle\beta|\phi_\alpha(y)\phi_\beta(x)|\alpha\rangle \right] \\
&= \frac{1}{a^2} \langle\beta| \left[\cancel{-\delta(x^0 - y^0) \partial_0 \phi_\beta(x) \phi_\alpha(y)} + \cancel{\delta(x^0 - y^0) \partial_0 \phi_\beta(x) \phi_\alpha(y)} + \right. \\
&\quad + \delta(x^0 - y^0) \partial_0 \phi_\beta(x) \phi_\alpha(y) + \theta(x^0 - y^0) \partial_0 \partial_0 \phi_\beta(x) \phi_\alpha(y) + \\
&\quad + 2 \frac{a'}{a} \left[\delta(x^0 - y^0) \phi_\beta(x) \phi_\alpha(y) + \theta(x^0 - y^0) \partial_0 \phi_\beta(x) \phi_\alpha(y) \right] + \\
&\quad + (k^2 + m^2 a^2) \left[\theta(x^0 - y^0) \phi_\beta(x) \phi_\alpha(y) \right] + \\
&\quad + \cancel{\delta(y^0 - x^0) \partial_0 \phi_\alpha(y) \phi_\beta(x)} - \cancel{\delta(y^0 - x^0) \partial_0 \phi_\alpha(y) \phi_\beta(x)} + \\
&\quad - \delta(y^0 - x^0) \phi_\alpha(y) \phi_\beta(x) + \theta(y^0 - x^0) \partial_0 \partial_0 \phi_\alpha(y) \phi_\beta(x) + \\
&\quad + \frac{a'}{a} \left[-\delta(y^0 - x^0) \phi_\alpha(y) \phi_\beta(x) + \theta(y^0 - x^0) \partial_0 \phi_\alpha(y) \phi_\beta(x) \right] + \\
&\quad \left. + (k^2 + m^2 a^2) \left[\theta(y^0 - x^0) \phi_\alpha(y) \phi_\beta(x) \right] \right| |\alpha\rangle
\end{aligned} \tag{2.48}$$

$$\tag{2.49}$$

$$\begin{aligned}
&= \frac{1}{a^2} \langle \beta | \left[\delta(x^0 - y^0) \partial_0 \phi_\beta(x) \phi_\alpha(y) + 2 \frac{a'}{a} \delta(x^0 - y^0) \phi_\beta(x) \phi_\alpha(y) + \right. \\
&\quad \left. - \delta(x^0 - y^0) \phi_\alpha(y) \partial_0 \phi_\beta(x) - 2 \frac{a'}{a} \delta(x^0 - y^0) \phi_\alpha(y) \phi_\beta(x) \right] | \alpha \rangle \quad (2.50)
\end{aligned}$$

$$= \frac{1}{a^2} \langle \beta | \left(\frac{1}{a^2} \delta(x^0 - y^0) [\pi_\beta(x), \phi_\alpha(y)] + 2 \frac{a'}{a} \delta(x^0 - y^0) [\phi_\beta(x), \phi_\alpha(y)] \right) | \alpha \rangle \quad (2.51)$$

$$= -i \frac{\langle \beta | \alpha \rangle}{a^4} \delta^4(x - y). \quad (2.52)$$

Similarly to the development from (2.28) to (2.31), in (2.49) some terms cancel each other out and two FLRW-KG equations were identified to 0. In the next step, the derivatives of the field (in respect only to x^0) can be expressed through its conjugate momentum by $\partial_0 \phi_\beta(x) = \frac{1}{a^2} \pi_\beta(x)$, as previously found through the Lagrangian. The antisymmetric commutator relations can be acknowledged to provide a tridimensional delta in the first term of (2.51), and 0 in the second one, culminating in the final form (2.52); the same one as the the appropriate Green's function's, apart from a simple normalization factor. It is crucial to point out that the vanishing of the second term is only possible on account of the multiplying temporal delta function forcing the commutator $[\phi_\beta(x), \phi_\alpha(y)]$ to be evaluated at the same time $x^0 = y^0$; if the mentioned delta was not present, the commutator of the fields could not be taken to 0, for the evolution of the field ϕ_β through time would render the operators it is composed of to have non-vanishing contributions with the ones from ϕ_α .

Although the same form of the Feynman propagator was used as in flat spacetime, is important to note that the states it makes reference to—namely $|\alpha\rangle$, $|\beta\rangle$, or the vacuum $|0\rangle$ —are not the same as states in the curved spacetime, for they are associated with distinct manifolds with different structures, and thus are defined in inherently separate Hilbert spaces. This renders the propagator in FLRW, despite its similarities with its counterpart, a different entity than D_F defined in (2.26); even if the states there were to be represented generally, instead of a choice of vacuum. Following this, the new FLRW-KG Feynman propagator is ultimately defined, in order to differentiate it between the old Minkowski one while also preserving its (appropriate) GF structure via normalization, by

$$G_{\alpha-\beta}(x, y) = \frac{\langle \beta | T \phi_\beta(x) \phi_\alpha(y) | \alpha \rangle}{\langle \beta | \alpha \rangle}. \quad (2.53)$$

Now the Klein-Gordon operator acting on the propagator reads

$$\Gamma G_{\alpha-\beta}(x, y) = -i \frac{1}{a^4} \delta^4(x - y). \quad (2.54)$$

The above equation (2.54) implies the probability of a transition happening between states is zero everywhere (and everywhen, since it is a four-dimensional delta on the right-hand side) except on a single event $x = y$. This was already the case for the Feynman propagator in Minkowski; but the FLRW propagator also takes the universe expansion into consideration. The achieved form (2.53) also makes physical sense, since in later stages of expanding universes, particles will be less localized due to the volume increase, making transitions in a single event $x = y$ less likely to happen; precisely by the inverse of the factor the universe has expanded by; which in the case at hand is reflected by the $\frac{1}{a^4(\eta)}$ factor that multiplies the delta function $\delta^4(x - y)$ that appears when the FLRW-KG operator acts on the propagator.

3 SCALAR FIELD SOLUTIONS IN EXPANDING UNIVERSES

Although the propagator allows for the comparison between states in different points in spacetime through the acting of fields onto the initial and final configurations, in order to complete said comparison it is necessary to know how the fields behave. It has been established that, in Minkowski, they possess the form (2.14), but since it does not solve the new FLRW-KG equation (2.39), their updated functional forms should be acquired. Furthermore, it would be interesting if exact solutions for the time component could be found, so that not only a field description would be fit, but also well understood predictions of their oscillations in different regimes could be made; this would result in observables that could be forecast, and even potentially tested by looking at the available data of regions of the universe, given careful observational analysis with a fitting modeling. Such solutions would also provide insight on the field's and propagator's behavior. The attention is consequently turned to the solving of the FLRW-KG equation (2.39).

3.1 Power Law and Universe Content

Many possible models can be proposed to the expansion of the universe; one of them, somewhat popular, frames the scale factor as a power law of time [49–51], not only being relatively simple to work with, but also being in accordance to accessible data [51]. Under these lights, the description here made concerns itself with a scale factor of expansion a being proportional—for the sake of simplicity, identical—to a power λ of the conformal time η , i.e.,

$$a(\eta) = \eta^\lambda. \quad (3.1)$$

For each power λ there is an equivalent equation of state w related to a fluid that serves as background for the content of the universe, three of the most studied are shown in Table 1.

Content	Equation of State	Scale Factor
Dust	$w = 0$	$a(\eta) = \eta^2$
Radiation	$w = \frac{1}{3}$	$a(\eta) = \eta$
Dark Energy	$w = -1$	$a(\eta) = \eta^{-1}$

Table 1 – Usual contents of the universe and their respective equation of state and scale factor dependency.

The power law framework actually follows from the Friedmann equations—which in the case at hand are given by

$$\left(\frac{a'}{a}\right)^2 = \frac{\kappa}{3}\rho a^2(\eta), \quad (3.2)$$

$$\frac{a''}{a} = \frac{\kappa}{6}(\rho + 3P), \quad (3.3)$$

where κ is Einstein's gravitational constant defined in terms of Newton's gravitational constant $\kappa = \frac{8\pi G}{c^4}$, P is the isotropic pressure of the fluid, and ρ its energy density—alongside the continuity equation:

$$\rho' + 3\frac{a'}{a}(\rho + P) = 0. \quad (3.4)$$

Assuming a barotropic fluid whose pressure depends linearly on the energy density, $P(\rho) = w\rho$, the continuity equation provides the relation

$$\rho(a) \propto a^{-3(1+w)}. \quad (3.5)$$

Plugging the above form in the first Friedmann equation (3.2), the power law structure appears:

$$a(\eta) = \left(\frac{\eta}{\eta_0} \right)^{\frac{2}{1+3w}}. \quad (3.6)$$

The constant η_0 is arbitrary and can be set to 1, in such a way that if η_0 were to represent present time, the universe would currently possess unit scale factor, $a(\eta_0) = 1$. The power can be regarded as a single variable λ , related to the equation of state by

$$\lambda = \frac{2}{1+3w}. \quad (3.7)$$

It is important to keep in mind that, although λ can be expressed in terms of w —and vice-versa—both are always constants in a given analysis, since they reflect a specific universe content associated with a fluid that bears a particular equation of state w and, consequently, a particular power λ .

3.2 Exact Solutions

With the parametrization (3.1), the FLRW-KG equation becomes

$$\varphi'' + 2\frac{\lambda}{\eta}\varphi' + \frac{1}{\eta^2}(k^2\eta^2 + m^2\eta^{2\lambda+2})\varphi = 0. \quad (3.8)$$

Here, instead of the previously used ϕ , chosen to represent the physical field, the slightly different letter φ is selected to represent the ordinary differential equation (ODE) solution. The physical field shall later be composed of the solutions found; hence the use of distinguishing the functions.

The above equation can be identified as one of the following form:

$$\varphi'' + \frac{p(\eta)}{\eta - \eta_0}\varphi' + \frac{q(\eta)}{(\eta - \eta_0)^2}\varphi = 0, \quad (3.9)$$

which can be approached by the Frobenius method around the point $\eta_0 = 0$, if the functions $p(\eta) = 2\lambda$ and $q(\eta) = k^2\eta^2 + m^2\eta^{2\lambda+2}$ are to respect some constraints.

In order to see if solutions are to be found using this method, the following form for the function is proposed:

$$\varphi = \sum_{n=0}^{\infty} \bar{\varphi}_n \eta^{n+r}. \quad (3.10)$$

By differentiating it, substituting in the ODE (3.8), and relabeling some indices, one gets

$$\left[\sum_{n=0}^{\infty} (n+r-1)(n+r)\bar{\varphi}_n + 2\lambda \sum_{n=0}^{\infty} (n+r)\bar{\varphi}_n + \sum_{n=2}^{\infty} k^2\bar{\varphi}_{n-2} + \sum_{n=2(\lambda+1)}^{\infty} m^2\bar{\varphi}_{n-2(\lambda+1)} \right] \eta^{n+r-2} = 0, \quad (3.11)$$

whose terms in brackets, since the equation is valid for all η , must sum up to 0 for each n individually. The power factor λ appearing in the summation index in the last term also indicates that solutions of this form are restricted to the cases where $2(\lambda+1) \in \mathbb{Z}$, limiting the range of λ to integers and half-integers. This restriction does not hinder the viability of the model, however, since backgrounds of interest in cosmological

studies usually have equations of states corresponding to integer powers of the scale factor—with the less common works reaching for half-integers [45], as it shall be discussed with more attention in chapter 4.

By analyzing equation (3.11) for $n=0$, the indicial polynomial is revealed:

$$r^2 + r(2\lambda - 1) = 0. \quad (3.12)$$

This, as a matter of fact, assumes the zeroth coefficient $\bar{\varphi}_0$ not to be zero. Since the subsequent coefficients will have a relation in term of a previous one, it is convenient to let the first one that appears to play such a role. The above is a simple quadratic equation and can be solved trivially, finding two roots:

$$r = \begin{cases} r_1 = 0, \\ r_2 = 1 - 2\lambda. \end{cases} \quad (3.13)$$

Studying the first root, $r_1 = 0$, and proceeding to the terms that appear when $n = 1$, one confronts the constraint $2\lambda \ ^1\bar{\varphi}_1 = 0$, where the superscript over the coefficient refers to the first root. This is respected if either the power λ or the the first-order-coefficient $\ ^1\bar{\varphi}_1$ is equal to zero. To keep the description general, the first coefficient is chosen to be the null one—besides, the power being zero simply describes a universe whose scale factor is constant, i.e., one with a Minkowski metric.

By continuing this process of individually increasing n and looking at the respective terms, it is possible to infer recurrence relations between the coefficients; namely, each coefficient $\ ^1\bar{\varphi}_n$ will depend on its predecessor by two instances $\ ^1\bar{\varphi}_{n-2}$. Since the coefficient of order one could and was chosen to be zero, this leads to the fact that all odd coefficients are also zero, leaving only the even ones, that respect the recurrence relation:

$$\ ^1\bar{\varphi}_n = -\frac{k^2}{n(n-1+2\lambda)} \ ^1\bar{\varphi}_{n-2}, \quad \text{while } n < 2(\lambda+1). \quad (3.14)$$

Of course, this relation will only be valid up until a certain point; until n reaches the critical value $2(\lambda+1)$, and the last sum starts contributing, after which the recurrence relation will become

$$\ ^1\bar{\varphi}_n = -\frac{1}{n(n-1+2\lambda)} (k^2 \ ^1\bar{\varphi}_{n-2} + m^2 \ ^1\bar{\varphi}_{n-2(\lambda+1)}), \quad \text{for } n \geq 2(\lambda+1). \quad (3.15)$$

Very similar calculations can be done for the second root $r_2 = 1 - 2\lambda$, providing the following relations¹:

$$\ ^2\bar{\varphi}_n = -\frac{k^2}{n(n+1-2\lambda)} \ ^2\bar{\varphi}_{n-2}, \quad \text{while } n < 2(\lambda+1), \quad (3.16)$$

$$\ ^2\bar{\varphi}_n = -\frac{1}{n(n+1-2\lambda)} (k^2 \ ^2\bar{\varphi}_{n-2} + m^2 \ ^2\bar{\varphi}_{n-2(\lambda+1)}), \quad \text{for } n \geq 2(\lambda+1). \quad (3.17)$$

It seems to be the case that two functions of the form (3.10), with r acquiring the values in (3.13) solve the ODE; however, there are some restrictions for them to be valid. The first one is that, in order for

¹ In the same manner that for the first root, when $n = 1$, a liberty of choice of which variable to be taken to zero fell upon the coefficient or a function of the power, for the second root the function of the power in question lead to a value of $\lambda = 1$, corresponding to a radiation background. Even though this is no longer a static universe, the coefficient can and is chosen to be null, so that generality is preserved as well as facilitating the Frobenius description by worrying only about the even coefficients.

the Frobenius method to be applicable, the ODE must not diverge faster than quadratically as $\eta \rightarrow 0$; so, lest the mass term in equation (3.8) does so, the power has a lower limit $\lambda \geq -1$. Secondly—hinted by the mass term summation index in equation (3.11)—the power λ is further restricted to certain values; namely, ones that respect $2(\lambda + 1) \in \mathbb{Z}$, i.e., the scale factor power can only be integer or half-integer. This renders the positive difference between the roots

$$N \equiv \begin{cases} r_1 - r_2 = 2\lambda - 1, & \text{for } r_1 > r_2, \\ r_2 - r_1 = 1 - 2\lambda, & \text{for } r_1 < r_2 \end{cases} \quad (3.18)$$

integer as well. If that were not the case, two Frobenius solutions would be guaranteed to exist, those being ${}^1\varphi$ and ${}^2\varphi$ in the form of (3.10) with the recurrence relations previously found for the coefficients; since N is always an integer, further investigation is required.

Carl Bender and Steven Orszag detailedly discuss the possible cases where Frobenius solutions do and do not exist in their book [68, Chapter 3]. Following their classification, the roots at hand fall into **Case II**. The only instance of **Case IIa** is given when the roots are the same, corresponding to $\lambda = \frac{1}{2}$; a stiff matter universe. In this case, there is only a single Frobenius solution, and other methods are required to find a second one—one is discussed shortly.

By expanding the functions $p(\eta)$ and $q(\eta)$ from (3.9) in Maclaurin series, $p(\eta) = \sum_{n=0}^{\infty} p_n \eta^n$ and $q(\eta) = \sum_{n=0}^{\infty} q_n \eta^n$, the next step is to analyze the equation:

$$0\bar{\varphi}_N = - \sum_{k=0}^{N-1} [(r+k)p_{N-k} + q_{N-k}] \bar{\varphi}_k, \quad (3.19)$$

where r is always the lower root. What must be checked is whether the right-hand side is truly zero, as the left-hand side clearly is; in other words, the above equation's consistency must be examined.

For $\lambda > \frac{1}{2}$, $N = r_1 - r_2 = 2\lambda - 1$, and the root r in equation (3.19) refers to r_2 . Substituting, the sum becomes:

$$\begin{aligned} 0 &\stackrel{?}{=} - \sum_{k=0}^{2\lambda-2} [(1-2\lambda+k)p_{2\lambda-1-k} + q_{2\lambda-1-k}] \bar{\varphi}_k & (3.20) \\ &= - \left\{ [(1-2\lambda)p_{2\lambda-1} + q_{2\lambda-1}] \bar{\varphi}_0 + [(-2\lambda+2)p_{2\lambda} + q_{2\lambda}] \bar{\varphi}_1 + \cdots + \right. \\ &\quad \left. + [(1-2\lambda+2\lambda-3)p_2 + q_2] \bar{\varphi}_{2\lambda-3} + [(1-2\lambda+2\lambda-2)p_1 + q_1] \bar{\varphi}_{2\lambda-2} \right\}. & (3.21) \end{aligned}$$

By comparing equations (3.8) and (3.9), it is noticeable that the only existing terms of $p(\eta)$ and $q(\eta)$ are $p_0 = 2\lambda$ for $p(\eta)$, and $q_2 = k^2$ and $q_{2\lambda+2} = m^2$ for $q(\eta)$. This, alongside with the odd coefficients not contributing, bring most terms of the sum to zero, reducing it to

$$0 \stackrel{?}{=} - \{-2q_2 \bar{\varphi}_{2\lambda-3}\} \quad (3.22)$$

$$= 2k^2 \bar{\varphi}_{2\lambda-3}. \quad (3.23)$$

For all $\lambda \in \mathbb{Z}$, the above quantity is indeed zero, since the coefficient $\bar{\varphi}_{2\lambda-3}$ would always be odd, falling into **Case IIbii**, implying two linearly independent Frobenius solutions do exist and were found earlier in this section. For half-integer powers, however, the above equation is not valid, since the coefficients would always be in terms of $\bar{\varphi}_0$, i.e., nonzero, falling into **Case IIbi**, where only one Frobenius solution exist, and other methods must be applied if one is to find a second linearly independent solution for the ODE.

For $\lambda < \frac{1}{2}$, $N = r_2 - r_1 = 1 - 2\lambda$, similar calculations lead to similar results:

$$0 \stackrel{?}{=} - \sum_{k=0}^{-2\lambda} [kp_{-2\lambda+1-k} + q_{-2\lambda+1-k}] \bar{\varphi}_k \quad (3.24)$$

$$= -k^2 \bar{\varphi}_{-2\lambda-1}. \quad (3.25)$$

Once again, if the power λ is an integer, the above equation is valid, and the solutions found apply; otherwise it is not and only one Frobenius solution exist. Taking into consideration the previous constraints found, this reduces to only two instances; $\lambda = -\frac{1}{2}$, which falls into **Case IIbi**; and $\lambda = -1$, which falls into **Case IIbii**.

The Frobenius method, therefore, provides two linearly independent solutions for integer powers, and one for half-integer ones. To find a second l.i. solution in the latter case, the reduction of order method using the Wronskian can be applied.

In it, the Wronskian of two solutions,

$$W({}^1\varphi, {}^2\varphi) \equiv {}^1\varphi {}^2\varphi' - {}^1\varphi' {}^2\varphi, \quad (3.26)$$

is differentiated through the solutions' derivatives following (3.8), yielding

$$W'({}^1\varphi, {}^2\varphi) = -2\frac{\lambda}{\eta} ({}^1\varphi {}^2\varphi' - {}^1\varphi' {}^2\varphi) \quad (3.27)$$

$$= -2\frac{\lambda}{\eta} W({}^1\varphi, {}^2\varphi). \quad (3.28)$$

Which provides a differential equation for the Wronskian itself, solved by

$$W({}^1\varphi, {}^2\varphi) = C_1 \eta^{-2\lambda}, \quad (3.29)$$

where C_1 is a constant.

By equating the definition (3.26) to the above form, the Wroskian condition that allows a second solution ${}^2\varphi$ to be found from a known first one ${}^1\varphi$ reveals itself:

$${}^2\varphi' = \frac{{}^1\varphi'}{{}^1\varphi} {}^2\varphi + \frac{C_1}{{}^1\varphi} \eta^{-2\lambda}. \quad (3.30)$$

Which is solved by

$${}^2\varphi = C_1 {}^1\varphi \int_{\eta_0}^{\eta} \frac{t^{-2\lambda}}{({}^1\varphi)^2} dt. \quad (3.31)$$

Although of a quadrature form, equation (3.31) provides a second solution to the FLRW-KG equation, linearly independent from the Frobenius one given by (3.10). They, alongside the recurrence relations (3.14)–(3.17), provide two modes ${}^1\varphi(\eta)$ and ${}^2\varphi(\eta)$ in closed forms that can be calculated with arbitrary precision for any time η in any FLRW universe modeled by a scale factor with an integer or half-integer power.

Summary

The scalar fields' equation of motion in FLRW universes whose scale factors respect a power law $a(\eta) = \eta^\lambda$,

$$\varphi'' + 2\frac{\lambda}{\eta}\varphi' + (k^2 + m^2\eta^{2\lambda})\varphi = 0, \quad (3.32)$$

can be solved in such a way to find two linearly independent solutions for all integer and half-integer powers above -1.

- Integer Powers:

For integer powers, $\lambda \in \mathbb{Z}$, the solutions are given by

$${}^1\varphi(\eta) = \sum_{n=0}^{\infty} {}^1\bar{\varphi}_n \eta^n, \quad (3.33)$$

$${}^2\varphi(\eta) = \sum_{n=0}^{\infty} {}^2\bar{\varphi}_n \eta^{n+(1-2\lambda)}, \quad (3.34)$$

where the coefficients for the first solution respect the recurrence relations:

$${}^1\bar{\varphi}_n = -\frac{k^2}{n(n-1+2\lambda)} {}^1\bar{\varphi}_{n-2}, \quad \text{while } n < 2(\lambda+1), \quad (3.35)$$

$${}^1\bar{\varphi}_n = -\frac{1}{n(n-1+2\lambda)} (k^2 {}^1\bar{\varphi}_{n-2} + m^2 {}^1\bar{\varphi}_{n-2(\lambda+1)}), \quad \text{for } n \geq 2(\lambda+1), \quad (3.36)$$

while the ones from the second solution follows

$${}^2\bar{\varphi}_n = -\frac{k^2}{n(n+1-2\lambda)} {}^2\bar{\varphi}_{n-2}, \quad \text{while } n < 2(\lambda+1), \quad (3.37)$$

$${}^2\bar{\varphi}_n = -\frac{1}{n(n+1-2\lambda)} (k^2 {}^2\bar{\varphi}_{n-2} + m^2 {}^2\bar{\varphi}_{n-2(\lambda+1)}), \quad \text{for } n \geq 2(\lambda+1). \quad (3.38)$$

- Half-Integer Powers:

For half-integer powers, $\lambda \in \mathbb{Z} + \frac{1}{2}$, the first solution is also of Frobenius form, while a linearly independent one is given in function of the first one

$${}^1\varphi(\eta) = \sum_{n=0}^{\infty} {}^1\bar{\varphi}_n \eta^n, \quad (3.39)$$

$${}^2\varphi(\eta) = C_1 {}^1\varphi \int_{\eta_0}^{\eta} \frac{t^{-2\lambda}}{{}^1\varphi^2} dt, \quad (3.40)$$

With the coefficients respecting the same recurrence relations:

$${}^1\bar{\varphi}_n = -\frac{k^2}{n(n-1+2\lambda)} {}^1\bar{\varphi}_{n-2}, \quad \text{while } n < 2(\lambda+1), \quad (3.41)$$

$${}^1\bar{\varphi}_n = -\frac{1}{n(n-1+2\lambda)} (k^2 {}^1\bar{\varphi}_{n-2} + m^2 {}^1\bar{\varphi}_{n-2(\lambda+1)}), \quad \text{for } n \geq 2(\lambda+1). \quad (3.42)$$

Whilst the second solution for half-integer powers (3.40) is not a power series as the first one, it is still useful to have such a clear formula, being linearly independent of its Frobenius counterpart. It, alongside the integer-power solutions (3.33)–(3.38), are novel results that solve the FLRW-KG equation in exact closed forms for a wide class of universes. Although not analytic—since they are not written as convergent power series, due to the recurrence being unsolved—they can be written as elementary functions and integrals of them. This is contrasting to the previously known solutions, which only existed in restrict cases [42, 43]—massless fields, for instance—as well as to numerical approximations provided by the equation of motion direct integration. These exact solutions may prove themselves faster to be calculated than numerical ones [46, 47], in addition to present a mathematical conciseness able to provide a structural understanding of the functions.

4 CONSEQUENCES OF TIME EVOLUTION

As it became evident in previous sections, geometry's dependence on time changes how the fields act, if they are to respect the physicality the Klein-Gordon equation brings to scalar fields. With solutions found for the FLRW-KG equation, it is possible to begin to probe how fields would behave in different ages of the universe, how they would be perceived by different frames of reference, as well as how to link those perceptions. Such merge of simultaneously quantum and relativistic natures give birth to surprising interpretations of physical effects, nearly ungraspable to existences so locally concentrated.

4.1 Different Observers and Bogolyubov Transformations

A given observer can describe excitations on the field from a vacuum, reducible to 0 by the annihilator $a_{\vec{p}}|0\rangle_\alpha = 0$, and whose basis for the Fock space rises from the creator $a_{\vec{p}}^\dagger|0\rangle_\alpha = |p\rangle_\alpha$. Just as easily, another frame of reference can be used to make descriptions, characterized by different operators and vacuum, $b_{\vec{p}}|0\rangle_\beta = 0$, $b_{\vec{p}}^\dagger|0\rangle_\beta = |p\rangle_\beta$. These two descriptions in terms of the scaled field $\chi(x) = a(\eta)\phi(x)$ read:

$$\chi_\alpha(\vec{x}, \eta) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \left\{ e^{i\vec{x}\cdot\vec{p}} v_p^*(\eta) a_{\vec{p}} + e^{-i\vec{x}\cdot\vec{p}} v_p(\eta) a_{\vec{p}}^\dagger \right\}, \quad (4.1)$$

$$\chi_\beta(\vec{x}, \eta) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \left\{ e^{i\vec{x}\cdot\vec{p}} u_p^*(\eta) b_{\vec{p}} + e^{-i\vec{x}\cdot\vec{p}} u_p(\eta) b_{\vec{p}}^\dagger \right\}, \quad (4.2)$$

where the subindices α and β inform about the observer that is describing said fields; each one related to its own vacuum and ladder operators as described above. Furthermore, the functions v and u play the role of Mukhanov-Sasaki modes that obey the equations

$$v_p'' + \omega_p^2(\eta) v_p = 0, \quad (4.3)$$

$$u_p'' + \omega_p^2(\eta) u_p = 0, \quad (4.4)$$

where the the frequency $\omega_p^2 = p^2 + m_{\text{eff}}^2(\eta)$ carries an effective mass term that will be shortly contextualized. In order to keep the field normalized, leading to normalized states as well as usual canonicity between ladder operators, the functions respect

$$\text{Im}(v_p' v_p^*) = \frac{v_p' v_p^* - v_p v_p'^*}{2i} = 1, \quad (4.5)$$

$$\text{Im}(u_p' u_p^*) = \frac{u_p' u_p^* - u_p u_p'^*}{2i} = 1. \quad (4.6)$$

They are related by

$$u_p(\eta) = \gamma_p v_p(\eta) + \delta_p v_p^*(\eta), \quad (4.7)$$

which respects the FLRW-KG solution, confirming the β description is just as valid as the α one. In the above equation, γ_p and δ_p are time-independent coefficients. Via the normalization conditions (4.5) and (4.6), the coefficients respect

$$|\gamma_p|^2 - |\delta_p|^2 = 1. \quad (4.8)$$

Although equations (4.1) and (4.2) are written through the descriptions of different observers, they are representations of the same field, thus they have to be equal, $\chi_\alpha = \chi_\beta$. This implies in the relations

between ladder operators sets:

$$a_{\bar{p}} = \gamma_p^* b_{\bar{p}} + \delta_p b_{-\bar{p}}^\dagger, \quad (4.9)$$

$$a_{\bar{p}}^\dagger = \gamma_p b_{\bar{p}}^\dagger + \delta_p^* b_{-\bar{p}}. \quad (4.10)$$

These are the *Bogolyubov transformations*, which relate the operators used by different observers to express field excitations along spacetime. They are equivalent to the relation between the functions

$$u_p(\eta) = \gamma_p v_p(\eta) + \delta_p v_p^*(\eta), \quad (4.11)$$

$$u_p'(\eta) = \gamma_p v_p'(\eta) + \delta_p v_p'^*(\eta). \quad (4.12)$$

With them, one finds the useful expression for the coefficients in terms of the Wronskian:

$$\gamma_p = \frac{W(u_p, v_p^*)}{2i}, \quad (4.13)$$

$$\delta_p = \frac{W(v_p, u_p)}{2i}. \quad (4.14)$$

As it was said, both sets of operators (a, a^\dagger) and (b, b^\dagger) can describe two sets of excited states,

$$|m_{\bar{p}_1}, n_{\bar{p}_2}, \dots\rangle_\alpha = \frac{1}{\sqrt{m_{\bar{p}_1}! n_{\bar{p}_2}! \dots}} \left[\left(a_{\bar{p}_1}^\dagger \right)^{m_{\bar{p}_1}} \left(a_{\bar{p}_2}^\dagger \right)^{n_{\bar{p}_2}} \dots \right] |0\rangle_\alpha, \quad (4.15)$$

$$|m_{\bar{p}_1}, n_{\bar{p}_2}, \dots\rangle_\beta = \frac{1}{\sqrt{m_{\bar{p}_1}! n_{\bar{p}_2}! \dots}} \left[\left(b_{\bar{p}_1}^\dagger \right)^{m_{\bar{p}_1}} \left(b_{\bar{p}_2}^\dagger \right)^{n_{\bar{p}_2}} \dots \right] |0\rangle_\beta, \quad (4.16)$$

in a respective manner. An arbitrary quantum state $|\psi\rangle$ can be written as a linear combination of these excitations:

$$|\psi\rangle = \sum_{m,n,\dots} C_{m,n,\dots}^\alpha |m_{\bar{p}_1}, n_{\bar{p}_2}, \dots\rangle_\alpha = \sum_{m,n,\dots} C_{m,n,\dots}^\beta |m_{\bar{p}_1}, n_{\bar{p}_2}, \dots\rangle_\beta. \quad (4.17)$$

If the frames are unitarily related, the probability of finding α -particles is $|C_{m,n,\dots}^\alpha|^2$, while the probability of finding β -particles is $|C_{m,n,\dots}^\beta|^2$. Even though α and β descriptions differ, the physical state they describe must be the same, regardless of their particle interpretation of it, since those are merely representations of the same reality; hence the equality.

By using one frame of reference's number operator to analyze another frame's vacuum state, for instance $N_{\bar{p}}^\alpha = a_{\bar{p}}^\dagger a_{\bar{p}}$ and $|0\rangle_\beta$, it is possible to calculate how many α -particles is present in the β -vacuum:

$${}_\beta \langle 0 | N_{\bar{p}}^\alpha | 0 \rangle_\beta = {}_\beta \langle 0 | a_{\bar{p}}^\dagger a_{\bar{p}} | 0 \rangle_\beta \quad (4.18)$$

$$= {}_\beta \langle 0 | \left(\gamma b_{\bar{p}}^\dagger + \delta^* b_{-\bar{p}} \right) \left(\gamma^* b_{\bar{p}} + \delta b_{-\bar{p}}^\dagger \right) | 0 \rangle_\beta \quad (4.19)$$

$$= {}_\beta \langle 0 | \left(|\gamma|^2 b_{\bar{p}}^\dagger b_{\bar{p}} + \gamma \delta b_{\bar{p}}^\dagger b_{-\bar{p}}^\dagger + \delta^* \gamma^* b_{-\bar{p}} b_{\bar{p}} + |\delta|^2 b_{-\bar{p}} b_{-\bar{p}}^\dagger \right) | 0 \rangle_\beta \quad (4.20)$$

$$= |\delta_p|^2 \delta^3(0). \quad (4.21)$$

Here the Bogolyubov transformations were used to write the ladder operators of the α -frame in terms of the β -frame, that results in most terms being canceled due to orthogonality, leaving only a term proportional to the norm squared of the Bogolyubov coefficient—not to be confused with the Dirac delta $\delta^3(0)$, that

represents an innocuous infinite, due to the infinite space taken into account. This calculation shows that a frame of reference using the set of operators $(a_{\vec{p}}, a_{\vec{p}}^\dagger)$ to describe excitations observes particles in states that another reference frame, which uses the set $(b_{\vec{p}}, b_{\vec{p}}^\dagger)$, does not. The exact amount observed for each momentum is dictated by the number density in terms of one of the Bogolyubov coefficient

$$n_p = |\delta_p|^2, \quad (4.22)$$

and the total mean density accounts for all of the possible momenta:

$$n = \int d^3\vec{p} |\delta_p|^2. \quad (4.23)$$

If the volume considered grows faster than $|\delta|^2$ decays, then the expected number of particles generated diverges and the physical interpretation is hindered; otherwise, n provides a reliable way to predict particle creation between different observers.

4.2 Choices of Vacua

As it has been mentioned throughout this work, the states associated with the ladder operators are linked to a specific frame of reference; one that observes a *vacuum*; a special state on which the entire spectrum is built. Nevertheless, what that special state signifies is not absolute. In Minkowski space, that notion is usually associated with a lack of particles, leaving the excitations only to be present due to field fluctuations, providing an invariant state between all inertial frames of reference; i.e., the class of states that are Lorentz invariant; this can conveniently be regarded as the minimum energy state, which all well behaved observers will agree upon. However in FLRW—and curved spacetimes in general, for that matter—even the minimum energy prescription will not result in a state concomitant to all observers, due to the system's lack of time translation symmetry; this results in what an observer perceives as containing no particles, another one—even that same frame of reference evolved in time—will perceive it as containing them, as it was seen in equations (4.18)–(4.21).

The nonuniqueness of the possible vacuum states leaves room to different prescriptions for how to define it, depending on the system in question. Due to this ambiguity a variety of vacua is used in current cosmology [69–75], and it is worth investigating some of them.

4.2.1 Instantaneous Lowest Energy Vacuum

Although it is not possible to globally define a state that bears the minimal possible energy for all points in spacetime, it is still possible to locally define an instantaneous lowest energy (ILE) state.

The action (2.37) in terms of the scaled field $\chi(x) = a(\eta)\phi(x)$ reads:

$$S = \frac{1}{2} \int d^3\vec{x} d\eta \left[\dot{\chi}^2 - (\nabla\chi)^2 - \left(m^2 a^2 - \frac{a''}{a} \right) \chi^2 \right], \quad (4.24)$$

with $\left(m^2 a^2 - \frac{a''}{a}\right) = m_{\text{eff}}^2$ playing the role of an effective field mass, that changes with time. This yields the Hamiltonian:

$$H(\eta) = \frac{1}{2} \int d^3 \vec{x} \left[\chi'^2 + (\nabla \chi)^2 + m_{\text{eff}}^2(\eta) \chi^2 \right]. \quad (4.25)$$

The Hamiltonian being time-dependent reflects the fact that there are no eigenvectors that could be used to define a global minimal energy state.

In order to find the initial conditions related to the lowest energy of an instant η_0 , that would define the instantaneous lowest energy vacuum $|0\rangle_{\eta_0}$, the expected energy value ${}_v \langle 0 | H | 0 \rangle_v$ needs to be calculated and then minimized in respect with the compact mode functions $v_p(\eta)$. To do so, the Hamiltonian is rewritten as

$$H(\eta) = \frac{1}{4} \int d^3 \vec{p} \left[a_{\vec{p}} a_{-\vec{p}} F_p^* + a_{\vec{p}}^\dagger a_{-\vec{p}}^\dagger F_p + \left(2a_{\vec{p}}^\dagger a_{\vec{p}} + \delta^3(0) \right) E_p \right], \quad (4.26)$$

by using the decomposed form for the field (4.1) and naming

$$E_p(\eta) \equiv |v_p'|^2 + \omega_p^2(\eta) |v_p|^2, \quad (4.27)$$

$$F_p(\eta) \equiv v_p'^2 + \omega_p^2(\eta) v_p^2. \quad (4.28)$$

The idea is to define a state that is brought to zero by the annihilator $a_{\vec{p}} |0\rangle_v = 0$ (as well as its dual counterpart being annihilated by the creator from the right ${}_v \langle 0 | a_{\vec{p}}^\dagger = 0$), in such a way that the expected energy value becomes

$${}_v \langle 0 | H(\eta_0) | 0 \rangle_v = \frac{1}{4} \delta^3(0) \int d^3 \vec{p} E_p(\eta_0). \quad (4.29)$$

Again, the divergence from the Dirac delta is harmless due to the infinite volume being taken into account; the more physical quantity would be the energy density:

$$\varepsilon(\eta_0) = \frac{1}{4} \int d^3 \vec{p} E_p(\eta_0) = \frac{1}{4} \int d^3 \vec{p} \left(|v_p'(\eta_0)|^2 + \omega_p^2(\eta_0) |v_p(\eta_0)|^2 \right). \quad (4.30)$$

The above equation's importance lies on the fact that, if values for v and v' are found in such a way that ε is minimized, those values would imply in initial conditions for the compact modes that minimize the energy (density) for the instant of time η_0 . This minimization must be realized for each momentum mode individually, leading to the minimization of the integrand

$$E_p(\eta_0) = |v_p'(\eta_0)|^2 + \omega_p^2(\eta_0) |v_p(\eta_0)|^2, \quad (4.31)$$

while also continuing to respect the normalization (4.5)

$$v_p'(\eta_0) v_p^*(\eta_0) - v_p(\eta_0) v_p^{*'}(\eta_0) = 2i. \quad (4.32)$$

Through the use of the ansatz $v_p = r_p e^{i\alpha_p}$, with r and α being real functions obeying $r_p^2 \alpha_p' = 1$, the integrand becomes

$$E_p(\eta_0) = r_p'^2 + r_p'^2 \alpha_p'^2 + \omega_p^2 r_p^2 = r_p'^2 \frac{1}{r_p^2} + \omega_p^2 r_p^2, \quad (4.33)$$

which is minimized when $r_p'(\eta_0) = 0$ and $r_p(\eta_0) = \sqrt{\omega_p(\eta_0)}$.

Thus, the initial conditions that determine the instantaneous lowest energy vacuum for a given instant of time η_0 are

$$v_p(\eta_0) = \frac{1}{\sqrt{\omega_p(\eta_0)}} e^{i\alpha_p(\eta_0)}, \quad (4.34)$$

$$v'_p(\eta_0) = i\sqrt{\omega_p(\eta_0)}e^{i\alpha_p(\eta_0)} = i\omega_p v_p(\eta_0). \quad (4.35)$$

Note that, although α remains to be determined, its specific value is irrelevant for the minimization, hence it can be set to zero with no complications arising. An important remark is that such a vacuum state is only valid for real frequencies $\omega_p^2 > 0$; if it were to have a complex part such that $\omega_p^2 < 0$, the energy would have no minimum, as could be seen in equation (4.33); in this case, the instantaneous lowest energy state does not exist, and it would be necessary to recur to another vacuum prescription.

Before investigating alternatives, however, it is instructive to check how this vacuum would behave if seen at a different, evolved, point in time. For these initial conditions $E_p(\eta_0)$ and $F_p(\eta_0)$ respectively assume the values $2\omega_p(\eta_0)$ and 0; in such a way that the Hamiltonian at that time is

$$H(\eta_0) = \int d^3\vec{p}\omega_p(\eta_0) \left[a_{\vec{p}}^\dagger a_{\vec{p}} + \frac{1}{2}\delta^3(0) \right]. \quad (4.36)$$

In order for the vacuum $|0\rangle_{\eta_0}$ to remain the lowest energy state, $F_p(\eta)$ would have to be zero for all η ; i.e. it would have to respect, for all η ,

$$F_p(\eta) = v_p'^2 + \omega_p^2(\eta)v_p^2 = 0. \quad (4.37)$$

The above ODE has the solution $v_p(\eta) = C\exp[\pm i\int\omega_p(\eta)d\eta]$, which does not satisfy (4.3), that characterizes the compact modes themselves, if the frequency varies with time. This ultimately means that the vacuum defined in η_0 will not hold for other times. It is possible, in fact, to calculate the energy that an evolved frame at time η_1 is expected to observe in the previously defined state, using the Bogolyubov transformations, calling b and b^\dagger the ladder operators in time η_1 :

$${}_{\eta_0}\langle 0|H(\eta_1)|0\rangle_{\eta_0} = {}_{\eta_0}\langle 0|\int d^3\vec{p}\omega_p(\eta_1) \left[b_{\vec{p}}^\dagger b_{\vec{p}} + \frac{1}{2}\delta^3(0) \right]|0\rangle_{\eta_0}, \quad (4.38)$$

$$= \delta^3(0) \int d^3\vec{p}\omega_p(\eta_1) \left[\frac{1}{2} + |\delta|^2 \right]. \quad (4.39)$$

The above equation makes it clear that, unless the Bogolyubov coefficient δ is equal to zero, in a time η_1 different from η_0 where the vacuum was defined, this energy is higher than the minimum possible value for that time, meaning someone at η_1 would detect particles in that same state.

4.2.2 Adiabatic Vacuum

Another useful vacuum prescription is one that describes particles in a slowly changing geometry. If the field's frequency of oscillation change slowly with time, the WKB approximation can be invoked to provide solutions to the linear equation of the compact modes (4.3). By proposing the form

$$v_p(\eta) = \frac{1}{\sqrt{W_p(\eta)}} \exp \left[i \int_{\eta_0}^{\eta} W_p(\eta) d\eta \right], \quad (4.40)$$

and applying it to the ODE, the following constraint for the frequency function emerges:

$$W_p^2 = \omega_p^2 - \frac{1}{2} \left[\frac{W_p''}{W_p} - \frac{3}{2} \left(\frac{W_p'}{W_p} \right)^2 \right]. \quad (4.41)$$

If $\omega_p(\eta)$ changes slowly—meaning the ratio between its change and its value becomes of relevant order, $\Delta\omega_p/\omega_p \sim 1$, only as the time analyzed goes to infinity compared to the characteristic time of the frequency, $T \gg 1/\omega_p$ —then equation (4.41) can be used as a recurrence relation allowing to find W_p in the

form of an asymptotic series. This means one can calculate the frequency function up to arbitrary order, with increasing accuracy up until some order N , after which the approximation becomes worse as more terms are considered. Substituting the function corrected up to N^{th} order, ${}^{(N)}W_p$, in the proposed form (4.40), the mode corrected up to the same order can be found, leading to the, ideally optimally accurate, initial conditions

$$v_p(\eta_0) = {}^{(N)}v_p(\eta_0), \quad (4.42)$$

$$v'_p(\eta_0) = {}^{(N)}v'_p(\eta_0). \quad (4.43)$$

This prescription can be viewed as an upgrade from the previously defined ILE. The WKB routine not only takes into account a single spatial sheet to define the vacuum—although it is entirely capable of limiting itself to that if desired—but also allows itself to make a particle description beyond its immediate neighborhood; this balance between a local notion of a particle—excitations verified by instruments in the lab—and a global one—corrections from those excitations coming from the universe expansion—renders the adiabatic vacuum useful when dealing with slowly expanding universes that causes fields' frequencies to evolve with time.

4.3 Solutions Behavior

Returning to the solutions found in Chapter 3, with the modes ${}^1\varphi$ (3.33) and ${}^2\varphi$ (3.34) in hand, the physical scalar field can be explicitly written as a linear combinations of them, alongside oscillatory exponentials and the creation and annihilation operators in momentum space:

$$\phi_\alpha(\vec{x}, \eta) = \int \frac{d^3\vec{p}}{(2\pi)^{\frac{3}{2}}} \left\{ e^{i\vec{x}\cdot\vec{p}} [\alpha_1 {}^1\varphi(\eta) + \alpha_2 {}^2\varphi(\eta)] a_{\vec{p}} + e^{-i\vec{x}\cdot\vec{p}} [\alpha_1 {}^1\varphi(\eta) + \alpha_2 {}^2\varphi(\eta)]^* a_{\vec{p}}^\dagger \right\}. \quad (4.44)$$

The coefficients $\alpha_{1,2}$ are tied to initial conditions of the modes, related to the choice of vacuum of the operators $a_{\vec{p}}, a_{\vec{p}}^\dagger$.

Both modes ${}^{1,2}\varphi$ were constructed in such a way that they individually solve the FLRW-KG equation. Their behavior—dependent on the mass m , the momentum k , and time η —can be plotted individually for each universe content—dictated by the power λ of the scale factor.

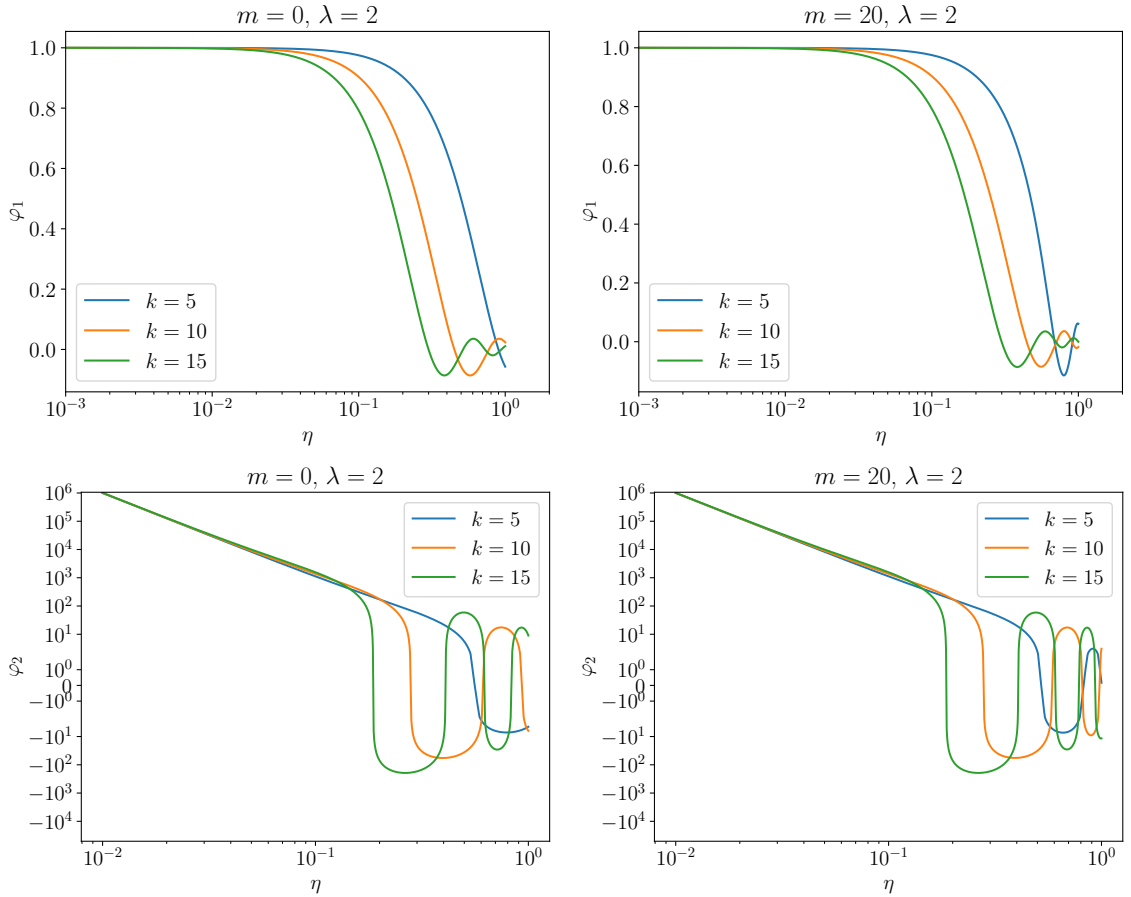


Figure 2 – Modes behavior for varying m and k as functions of conformal time η in dust dominated universes.

Illustratively, Figure 2 shows how ${}^1\varphi$ and ${}^2\varphi$ behave as time goes on for varying parameter choices in dust dominated universes ($\lambda = 2$). It is clear that the mode ${}^1\varphi$ (first row) begins frozen in a super-Hubble phase, and as time goes on it starts to oscillate and acquires a wave-like nature around the point at which η reaches values between 10^{-1} and 1. It is also noticeable that the higher the momentum k , the sooner the field starts to oscillate, while the the mass m has the effect of tuning how fast the oscillations happen once they start. The effects of both these parameters are in accordance to expectations; the FLRW-KG equation

(3.8) displays k coupled to the square of the conformal time, while m , for positive integer λ , is coupled to a higher power, meaning as η increases, the more massive field will always oscillate faster.

The observations mentioned above can also be made for the second mode ${}^2\varphi$ (second row), except it does not start frozen, but instead diverges the closer it is analyzed to $\eta = 0$. The initial non-oscillatory behavior indicates the lack of wave-like interaction the scalar particle could have as $\eta \rightarrow 0$; this leads to the horizon problem, where the universe content does not have the opportunity to homogenize itself in its earlier stages, requiring further complexification in the modeling process—such as the implementation of bouncing or inflationary phases [76–78]. After the mode begins to oscillate, however, the wave description is fit, with the same afore mentioned effects of k and m .

Despite the second mode in the case of half-integers λ coming not from the Frobenius approach, it does provide similar behavior to the previously shown. Figure 3 presents the mode originated by the quadrature solution’s evolution. The effects of the momentum k and mass m remain the same, while in the super-Hubble phase its value diverges as well, albeit in a much more contained manner than its exact counterpart.

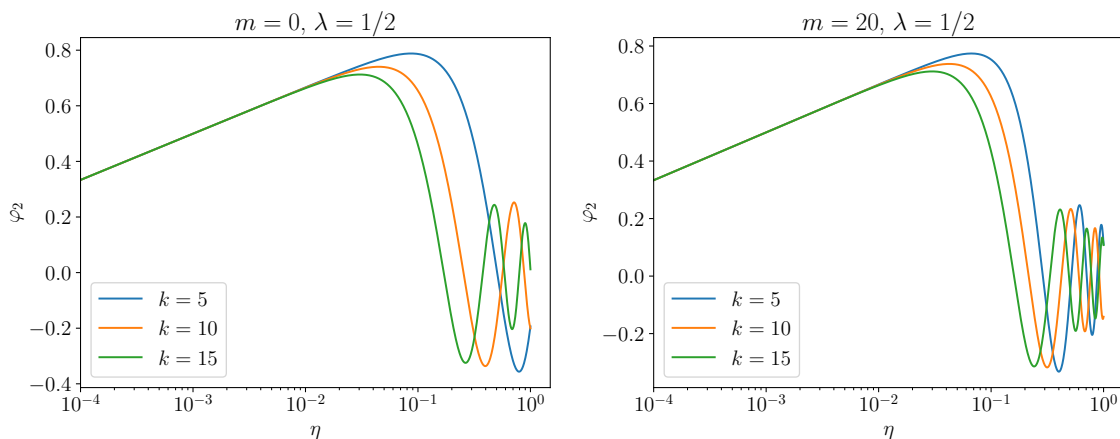


Figure 3 – Second mode behavior for varying m and k as functions of conformal time η in stiff-matter dominated universes.

4.3.1 Particle Creation

The modes also enable the prediction of particle creation between observers separated by the universe expansion, using the Bogolyubov transformations. As it was discussed in section 4.1, observers at different points of spacetime have different notions of particles, and the state one sees as containing none, another one can see as excited; this holds even if the second observer is one that once was the first observer, only evolved in time. The described phenomenon can be quantified by using the Bogolyubov coefficient δ , as in (4.22).

By composing the modes into

$$\zeta \equiv {}^1\varphi + i {}^2\varphi, \quad (4.45)$$

$$\zeta^* \equiv {}^1\varphi - i {}^2\varphi, \quad (4.46)$$

the function u an observer uses to describe field excitations at an arbitrary instant η

$$u(\eta) = c_1\zeta(\eta) + c_2\zeta^*(\eta), \quad (4.47)$$

can be related to the function an observer uses at a different time $\eta_0 = \eta - \Delta\eta$

$$v(\eta_0) = d_1\zeta(\eta_0) + d_2\zeta^*(\eta_0), \quad (4.48)$$

by equation (4.7)¹. The Bogolyubov coefficient that informs about particle creation can be written in terms of the Wronskian of the functions, as in (4.14); returning

$$\begin{aligned} \delta(\eta, \eta_0) = \frac{1}{2i} [& d_1c_1W(\zeta(\eta), \zeta(\eta_0)) + d_1c_2W(\zeta^*(\eta), \zeta(\eta_0)) + \\ & + d_2c_1W(\zeta(\eta), \zeta^*(\eta_0)) + d_2c_2W(\zeta^*(\eta), \zeta^*(\eta_0))]. \end{aligned} \quad (4.49)$$

Simply taking the conjugate of $\delta(\eta, \eta_0)$ and multiplying by it, the expected particle density created between two frames separated by time, $|\delta(\eta, \eta_0)|^2$, is acquired.

Having chosen initial conditions, which will be reflected by the values of the coefficients, the particle density can be plotted by varying parameters, much alike the modes themselves.

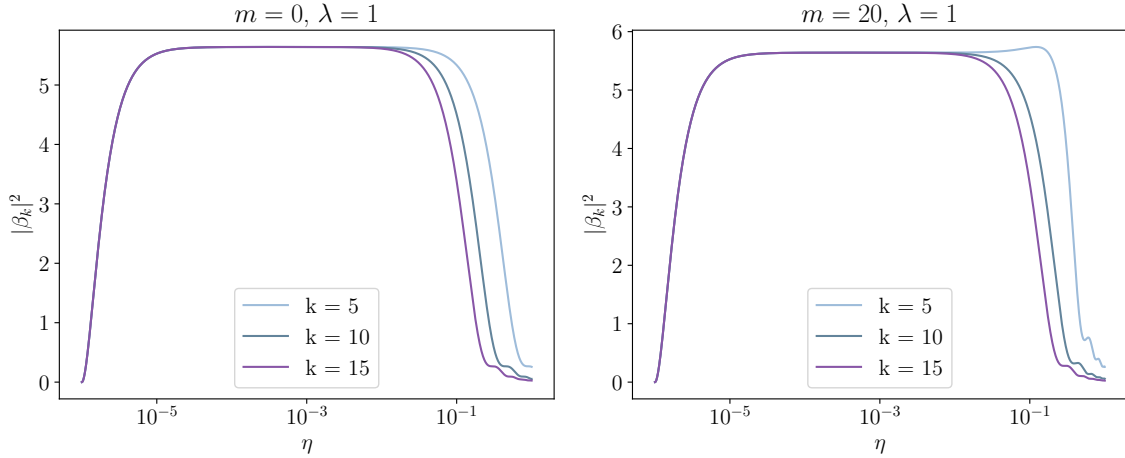


Figure 4 – Expected particle density behavior for varying m and k as functions of conformal time η in radiation dominated universes.

Figure 4 demonstrates, for radiation dominated universes, particle density rising as time passes, then plateauing for a while before beginning to oscillate while quickly decreasing to zero again. Once more, increasing k has the effect of making the oscillations begin earlier, while the mass makes them happen faster; notably, the right-most curve in the second graph has a "shoulder", implying particle creation can still increase after the plateau, just before quickly decreasing. Separating the Bogolyubov coefficient in real and imaginary parts, it is mayhap surprising to find that the initial rise comes entirely from the real part, as it can be seen in Figure 5.

The second graph of Figure 5 is plotted so the behavior closer to $\eta = 1$ can be seen without too much visual noise. In this region the oscillating behavior is quite akin to the modes', plotted in Figure 2 and Figure 3, nevertheless, the physical meaning is drawn from the multiplication of real and imaginary parts, instead of them separately, leaving a less clean pattern os oscillations. Remarkably, however, the expected particle density begins to stabilize around zero, which means that, even if an evolved observer starts seeing a considerable amount of particle being generated—up until just before $\eta \sim 10^{-1}$ —it quickly stops perceiving

¹ Remember η_0 can be written in terms of η .

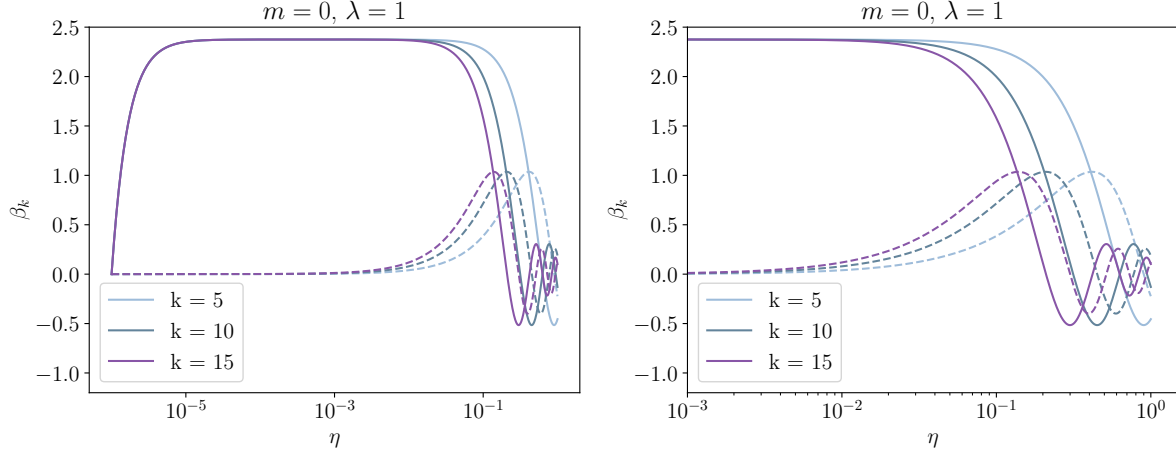


Figure 5 – Real (continuous lines) and imaginary (dashed lines) parts of Bogolyubov coefficient for varying k in radiation dominated universes.

them, and the state it observes is much more alike the initial one, with not many particles being generated in comparison to the observer at a previous time η_0 . Interestingly enough, this happens when the wave-like behavior of the modes begin; thus, as far as the particle interpretation of fields is valid, the particle creation between a reference frame and another evolved in time is minor, although fundamentally not zero.

4.3.2 The General Propagator

With the physical field (4.44), the general propagator, up until this point given by (2.53), can be explicitly written as

$$\begin{aligned}
 G_{\alpha-\beta} = & \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{d^3q}{(2\pi)^{\frac{3}{2}}} \frac{\langle \beta | b_p a_q^\dagger | \alpha \rangle}{\langle \beta | \alpha \rangle} \\
 & \cdot \left\{ \theta(\eta_2 - \eta_1) e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} [\beta_1 {}^1\varphi(\eta_1) + \beta_2 {}^2\varphi(\eta_1)] [\alpha_1 {}^1\varphi(\eta_2) + \alpha_2 {}^2\varphi(\eta_2)]^* \right. \\
 & \left. + \theta(\eta_1 - \eta_2) e^{-i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} [\beta_1 {}^1\varphi(\eta_2) + \beta_2 {}^2\varphi(\eta_2)] [\alpha_1 {}^1\varphi(\eta_1) + \alpha_2 {}^2\varphi(\eta_1)]^* \right\}. \quad (4.50)
 \end{aligned}$$

This form is quite comprehensive, encompassing a vast array of scenarios with general initial and final states $|\alpha\rangle$ and $|\beta\rangle$, associated with their respective ladder operators (a_q, a_q^\dagger) and (b_p, b_p^\dagger) ; the modes ${}^{1,2}\varphi$ are to be set for individual background choices, and the coefficients $\alpha_{1,2}$ and $\beta_{1,2}$ reflect the vacua; all of the quantities can be substituted once a scenario is established; thus, the probability amplitude of a configuration of scalar particles to another can be computed for any two points x and y in an expanding universe.

5 CONCLUSION

This thesis began by modeling a free massive real scalar field immersed in a FLRW universe background, that expands following a power law of conformal time, $a(\eta) = \eta^\lambda$; under these conditions the FLRW-KG equation—the appropriate scalar field equation of motion—was found and solved using the Frobenius method and reduction of order. The cases solved could be divided into two instances; for integer powers, $\lambda \in \mathbb{Z}$, two exact, linearly independent, solutions were found; for half-integer powers, $\lambda \in \mathbb{Z} + \frac{1}{2}$, it was obtained one exact and one quadrature solution. With the closed forms for them in hand, their evolution in time were cast, and it was observed that they do present oscillatory behavior—characteristic of the field description—for a period of the expansion onward. Furthermore, the modes allowed for the prediction of particle creation, linking two different vacuum states separated by the evolution of the universe. Finally, the Feynman propagator for scalar fields in FLRW metrics was explicated in its most general form, allowing calculations to be performed using the Dyson series and Feynman diagrams formalism.

The field framework, standing as a powerful description of nature in its most fundamental scales, was shown to interact with spacetime geometry in interesting ways, presenting behaviors absent in the Quantum Field Theory or General Relativity alone. This thesis explored such interplay, in search of bringing clarity to the way scalar particles behave in universes akin to the observed, taking its expansion into account. As far as the field’s evolution throughout different ages of the universe is concerned, such enlightenment was achieved; with the obtainment of novel exact solutions for the scalar modes; as well as regarding the exploration of particle generation. These quantities were discussed and evaluated as the universe expansion played out, unifying the field description with cosmological scales. Moreover, they allow the practical implementation of the propagator in a myriad of calculations, permitting the extension of QFT formalism onto curved spacetimes.

This work made use of the Frobenius method to find previously uncovered solutions to the FLRW-KG equation; the method, however, as any other, has its limitations; the most evident one is the range it covers, it being the restriction for the power λ to the integers and half-integers above -1. Another limitation of the results achieved is the recurrence relations for the Frobenius solutions that remain unsolved; this could be a very interesting mathematical challenge to tackle, since—different from the previous restriction, has not been concluded to be irresolvable. The Frobenius technique does not provide a second linearly-independent solution to the half-integers case, but it does not mean **no** linearly-independent exact solution exists; if it does and one is found, it would enhance the usefulness of the scalar modes, as far as computational efficiency is concerned, as well as leaving the solutions cleaner, enhancing their behavior’s clarity.

Aside from these individual limitations, further generalizations could also be applied. The choice of FLRW metrics, for instance, was chosen based on its simplicity and its relevance in cosmological studies, but there are no reasons why other geometries should not be approached in future works. Moreover, the theory could be generalized to encompass more complex fields—spinors, for instance, would allow the modeling of electrons; or gauge bosons, necessary for a complete electromagnetic description; or even tensorial particles, favored by the graviton representation—and, having their propagators similarly encountered, interactions between different particles could be studied, modeling richer and more realistic universes.

Understanding the interplay between Quantum Field Theory and curved spacetimes remains a cornerstone challenge in theoretical physics; the road of possible discoveries yet to be made are vast and untouched, and the results here presented provide a small but hopefully meaningful step in that broader pursuit.

Appendix

APPENDIX A – VOLUME FORM

In the context of differential geometry, integrands are properly understood as n-forms. In order to define the volume element on manifolds, their structure should be implemented in its construction. One could then begin with the antisymmetric wedge products of the n bases:

$$d^n x = dx^0 \wedge \dots \wedge dx^{n-1}. \quad (\text{A.1})$$

$$= \frac{1}{n!} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}. \quad (\text{A.2})$$

This is a coordinate-dependent quantity, and given the transformation between two frames of references x^μ and $x^{\mu'}$

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu, \quad (\text{A.3})$$

the element transforms as

$$\frac{1}{n!} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} = \frac{1}{n!} \epsilon_{\mu_1 \dots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}} \dots \frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}} dx^{\mu'_1} \wedge \dots \wedge dx^{\mu'_n} \quad (\text{A.4})$$

$$= \frac{1}{n!} \left| \frac{\partial x^\mu}{\partial x^{\mu'}} \right| \epsilon_{\mu'_1 \dots \mu'_n} dx^{\mu'_1} \wedge \dots \wedge dx^{\mu'_n}. \quad (\text{A.5})$$

Although the element is not invariant, the Levi-Civita tensor ϵ is, hence the primed indices from (A.4) to (A.5).

The only thing preventing the proposed construction (A.1) from remaining invariant under coordinates change is the term $\left| \frac{\partial x^\mu}{\partial x^{\mu'}} \right|$, which can be completely compensated by multiplying by the square root of the absolute value of the determinant of the space's metric the coordinates are embedded in, $\sqrt{|g|} = \left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right|$. Thus, the volume element that permits integration to be performed in manifolds with geometry associated with the metric $g_{..}$ is the volume form:

$$\sqrt{|g|} d^n x \equiv \sqrt{|g|} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}. \quad (\text{A.6})$$

And the integral of a scalar function $f(x)$ over a n-dimensional manifold is given by:

$$I = \int f(x) \sqrt{|g|} d^n x. \quad (\text{A.7})$$

APPENDIX B – GENERAL COUPLING SOLUTIONS

By including a term proportional to the Ricci scalar R , coupled by the coupling constant ξ , the general action reads

$$S = \frac{1}{2} \int \sqrt{|g|} d^4x (g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - m^2 \phi^2 - \xi R \phi^2). \quad (\text{B.1})$$

In the case of interest—assuming the mostly minus signature, spatially flat sheets and the scale factor as a power law—the generally coupled FLRW-KG equation follows from the action:

$$\phi'' + 2 \frac{\lambda}{\eta} \phi' + \left[k^2 + m^2 \eta^{2\lambda} - 6\xi \frac{\lambda(\lambda-1)}{\eta^2} \right] \phi = 0. \quad (\text{B.2})$$

The Frobenius method proposes solutions in the form of power series of η :

$$\varphi(\eta) = \sum_{n=0}^{\infty} \bar{\varphi}_n \eta^{n+r}. \quad (\text{B.3})$$

where the indicial polynomial provides two roots r :

$$r_{1,2} = \frac{1}{2} \left[1 - 2\lambda \pm \sqrt{(2\lambda-1)^2 + 24\xi\lambda(\lambda-1)} \right]. \quad (\text{B.4})$$

The procedure also supplies recurrence relations for the coefficients $\bar{\varphi}_n$, which remain the same up until a critical $n < n_c \equiv 2(\lambda+1)$, after which point they acquire a new form, implementing the mass term:

$${}^{1,2}\bar{\varphi}_n = -\frac{k^2 {}^{1,2}\bar{\varphi}_{n-2}}{(n+r_{1,2})(n+r_{1,2}+2\lambda-1-6\xi\lambda(\lambda-1))}, \quad \text{while } n < 2(\lambda+1), \quad (\text{B.5})$$

$${}^{1,2}\bar{\varphi}_n = -\frac{k^2 {}^{1,2}\bar{\varphi}_{n-2} + m^2 {}^{1,2}\bar{\varphi}_{n-2(\lambda+1)}}{(n+r_{1,2})(n+r_{1,2}+2\lambda-1-6\xi\lambda(\lambda-1))}, \quad \text{while } n \geq 2(\lambda+1). \quad (\text{B.6})$$

In the minimally-coupled case, $\xi = 0$, equation (B.4) returns the roots

$$r_1 = 0, \quad (\text{B.7})$$

$$r_2 = 1 - 2\lambda, \quad (\text{B.8})$$

and the relations reduce themselves to the previously mentioned (3.35)–(3.38).

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