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EDUARDO JOSÉ BARROSO

**BOSONIZATION APPROACH FOR THE QUANTUM HALL  
EFFECT**

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Londrina  
2020

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EFFECT**

A thesis submitted in fulfillment of the requirements  
for the degree of Masters in the Physics Department

Advisor: Prof. Dra. Paula Fernanda Bienzobaz  
Co-Advisor: Prof. Dr. Pedro Rogerio Sergi Gomes

Londrina  
2020

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## RESUMO

Neste trabalho, estudamos como aplicar bosonização ao efeito Hall quântico. A bosonização consiste em descrever uma teoria quântica de campos de férmions em termos de bósons. Provamos que os sistemas bosonizado e original levam às mesmas funções de correlação, comprovando o sucesso da técnica. Estudamos alguns sistemas, como o modelo de Thirring e o modelo de Schwinger, que podem se tornar mais fáceis de resolver após o uso da bosonização. Em seguida, prosseguimos com o estudo do efeito Hall quântico, caracterizado por valores quantizados da condutância de Hall os quais não são esperados pela teoria clássica. Utilizamos a abordagem microscópica e a abordagem efetiva, descrita pela teoria de Chern-Simons, para estudar o efeito Hall quântico inteiro e o efeito Hall quântico fracionário. Em seguida, descrevemos o sistema Hall como uma matriz de fios quânticos sem spin em  $1 + 1$  dimensões e aplicamos bosonização no modelo. Observamos que o sistema bosonizado captura a física do efeito Hall quântico, bem como as características topológicas da teoria de Chern-Simons, o que leva a uma conexão direta entre os fios quânticos e a abordagem efetiva. Essa conexão gera um vínculo entre os graus de liberdade microscópicos e a teoria efetiva.

**Palavras-chave:** Bosonização. Teoria quântica de campos. Fios quânticos. Chern-simons.

## ABSTRACT

In this work, we study how to apply bosonization to the quantum Hall effect. Bosonization consists in describing a quantum field theory of fermions in terms of bosons. We prove that both the bosonized and the original systems lead to the same correlation functions, proving the success of the technique. We study a few systems, such as the Thirring model and the Schwinger model, that can become easier to solve after using bosonization. We then proceed with the study of the quantum Hall effect, characterized by quantized values of the Hall conductance that are not expected by the classical theory. We use the microscopic approach and the effective approach, described by the Chern-Simons theory, to study the integer quantum Hall effect and the fractional quantum Hall effect. Next, we describe the Hall system as an array of spinless quantum wires in  $1 + 1$  dimensions and apply bosonization to the model. We observe that the bosonized system captures the quantum Hall effect physics, as well as the topological features of the Chern-Simons theory, which leads to a direct connection between the quantum wires and the effective approach. This connection generates a link between the microscopic degrees of freedom and effective theory.

**Keywords:** Bosonization. Quantum field theory. Quantum wires. Chern-simons.

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## 1 INTRODUCTION

In physics, systems can be classified as strongly or weakly coupled, regarding the interactions present in the systems. These interactions dictate how we must approach the system and what tools may be required to study the problem. When the interactions are small enough, a common approach is to use perturbation theory [1], while for a strongly coupled system, there is no systematic analytical method. In this thesis, we study the bosonization technique, a tool in quantum field theory that is used to describe Dirac fermions by a bosonic system in 1+1 dimensions [2, 3, 4]. With bosonization, we observe that a strong coupled fermionic system may be described as a weakly coupled or even as a non-interacting bosonic system.

Bosonization is a kind of duality where two different theories turn out to describe the same physics [4]. To properly describe the fermionic by the bosonic theory, we need to prove that both systems capture the same physics. Even though fermions and bosons have different statistics [5], we can prove the effectiveness of this technique when we compute the correlation functions of the transformed and non-transformed systems and find the same result. The bosonization technique is proven to work best in (1+1) dimensions, where there is a map between the fermionic and the bosonic operators [3]. Some systems, such as the Thirring and the Schwinger models are easier to study with bosonization and are important examples of this technique [6, 7].

A physical system of great interest where perturbation theory may not be used is the Quantum Hall system. It is constituted of charged electrons restricted to move in a two-dimensional plane in the presence of an external magnetic field [8]. There are characteristics present in the Hall system that would not be expected by classical physics and have been observed experimentally, such as the plateaux structure of the Hall resistivity. When we consider interactions between the electrons in the system, some features indicate that the system becomes strongly coupled, for instance, the presence of quasi-particles excitations with fractional charge and statistics [9]. Therefore, we cannot use perturbation theory in the system, which makes it difficult to study the problem.

The quantum Hall effect can be understood either by a variational wave function, the Laughlin wave function [10], or by an effective quantum field theory, the Chern-Simons theory [11]. In the microscopic approach, we begin with an electronic Hamiltonian, using usual quantum mechanics [12, 13]. However, in the presence of interaction between the electrons, the system becomes strongly coupled and we cannot use perturbation theory. This implies that the fully microscopic theory is not approachable for any analytical method. The Laughlin wave function is not an exact solution of the Schrödinger equation of the system, although it is a satisfactory approximate solution for the microscopic approach.

In the effective field theory approach, we construct a low-energy effective

action capturing the general properties of the Hall system [14, 8, 9]. The effective theory approach allows us to study the system from a macroscopic point of view. However, this approach does not connect the effective action with the microscopic degrees of freedom, which would be useful for a deeper understanding. To make this connection, we use the bosonization technique to study the quantum Hall system through the so-called quantum wires [15, 16, 17]. In this approach, we describe the Hall fluid as an array of coupled spinless fermions in 1+1 dimensional wires. The dimension of the wires enables the bosonization technique to be implemented straightforwardly, as there is a map between the fermionic and bosonic operators in 1+1 dimensions.

The wires approach may be seen as a bridge between the microscopic and the effective approach [18], as it starts with electronic degrees of freedom and ends up describing some features of the quantum Hall system in the same way described by the effective theory. Therefore, we may search for a connection between the wires and the effective Chern-Simons theory [18, 19].

The central objective of this work is to apply the bosonization technique to the quantum Hall effect in the framework of the quantum wires approach and examine the connection with the effective Chern-Simons theory. The bosonization technique allows us to understand properties of the system in the strong coupling regime, which was not possible in the initial microscopic approach. Also, the quantum wires approach leads to a direct connection between the microscopic and effective theories, a feature that was absent in the other approaches.

The study begins in chapter 2, which is devoted to reviewing some basic concepts of quantum field theory and the renormalization group, concepts that will be needed for the construction of this thesis. In chapter 3, we study the fermionic and bosonic systems in (1+1) dimensions and then describe the bosonization technique. We give some examples of the bosonization framework and its applications in chapter 4. In chapter 5, we study the Quantum Hall Effect in the microscopic approach and the effective theory approach. In chapter 6 we apply the bosonization technique to the wires system and use the results to make a connection with the effective Chern-Simons theory, in chapter 7. We then conclude this work with some final remarks in chapter 8.

## 2 BASIC TOOLS

This chapter is devoted to review some basic tools that are needed for the development of this work. Before getting started we should define some notations that are going to be used in this entire thesis. The metric used here is  $(+,-,-,-)$  and  $c = \hbar = e = 1$ . A parameter without arrow, as  $p$  or  $x$ , stands for the 4-component vector while a parameter such as  $\vec{p}$  represents spacial vectors. Also, variables with Latin alphabet index such as  $p_i$  represent the spacial vector components and variables with greek letters such as  $p_\mu$  represent the 4-vector components. And last, an integral without limits means that we are taking it to be  $\int_{-\infty}^{\infty}$ . We want to approach two main topics, canonical quantization in quantum field theory and renormalization.

### 2.1 QUANTUM FIELD THEORY

Quantum mechanics equations are usually taken with non-relativistic particles [5]. What would happen if we considered the relativistic limit? There is actually a relativistic quantum mechanics, but it is quite restrictive and do not explain some concepts, as the creation of the pair particle anti-particle. In this section we go trough the basic concepts of Quantum Field Theory (QFT), which is the result of quantum mechanics and special relativity [20].

#### 2.1.1 Klein-Gordon Equation

To acquire some familiarity, we first consider a basic relativistic equation, the Einstein energy-momentum relation

$$E^2 - P^2 = p^\mu p_\mu = m^2, \quad (2.1)$$

where  $p^\mu$  is the 4-momentum vector,  $E$  is the energy and  $m$  is the mass. We can now promote these variables to operators and apply equation (2.1) to a wave function represented by the wave function  $\phi$ . Using the usual rules of quantum mechanics, the equation becomes

$$\begin{aligned} p^\mu p_\mu \phi &= m^2 \phi, \\ (i\partial^\mu)(i\partial_\mu)\phi &= m^2 \phi, \\ (\square + m^2)\phi &= 0. \end{aligned} \quad (2.2)$$

Here,  $\square \equiv \partial^\mu \partial_\mu$ . Equation (2.2) is known as the Klein-Gordon equation. It gives the dynamics of the particle and has a relativistic character, since we are considering time and space to be at the same level, and can be solved by a plane wave solution ( $\phi(x) = e^{\pm ik \cdot x}$ ). The problem is that if we look at the energy spectrum of this solution, we would get negative energies, which are not allowed [20]. Also, a negative energy results in negative probability density to find the

particle in a given state. This is known as the Klein-Gordon paradox. In this sense, the Klein-Gordon equation cannot be applied to a single particle wave function [5]. Before jumping into conclusions, let us take a closer look at another equation.

### 2.1.2 Dirac Equation

In the previous subsection, we used the Einstein relation to get an equation that relates momentum, mass, and energy by quadratic terms. As we have in the Schrödinger equation, here we also want a linear relation between momentum and energy. Let us consider equation (2.1), that is,

$$p^\mu p_\mu = m^2. \quad (2.3)$$

Allowing this to be a matrix relation in 3+1 dimensions, there exists four linear independent matrices  $\gamma^\mu$  such that

$$\not{p} \equiv \gamma^\mu p_\mu, \quad (2.4)$$

with  $\mu = 0, 1, 2, 3$ . With some analysis, it is stated that if this four matrices satisfy the Clifford algebra and some other properties, it is possible to obtain a linear relation between energy and momentum with them [5, 1]. These matrices are called the Dirac matrices. In 1+1 dimensions, we can use two Pauli matrices, as we will see in the next chapters.

We shall take the square root of equation (2.3) in order to get a linear relation. By doing so and applying it to a wave function  $\psi$ , we get the Dirac equation given by

$$(\gamma^\mu p_\mu - m)\psi = 0. \quad (2.5)$$

In coordinate space it becomes

$$(i\gamma^\mu \partial_\mu - m)\psi = (i\not{\partial} - m)\psi, \quad (2.6)$$

where  $\gamma^\mu \partial_\mu \equiv \not{\partial}$ . The plane-wave solution for this theory involves negative energies. Somehow we get the sense that a theory involving quantum mechanics and relativity must involve a many particle theory, and that is how quantum field theory arises [5].

In QFT we combine quantum theory and the principle of relativity, which results in the existence of fields [20]. In the next sections we promote our previous variables  $\phi$  and  $\psi$  to fields operators. In the same way that we consider time in a parameter to quantize  $x$  and  $p$  in quantum mechanics, now we promote the momentum and position to parameters to quantize our new fields operators, that now represent a many-particle operator. This method is called second quantization [21]. In quantum field theory, we can interpret the particles we had in quantum mechanics as being excitations of these fields [1]. The Klein-Gordon equation is

defined for scalar fields, while the Dirac equation is defined for spinorial fields [13]. The next sections are devoted for the solution of these equations. A deeper development can be found in references [1, 5, 20].

### 2.1.3 Klein-Gordon Field Theory

We now quantize our fields in the canonical scheme, as we are used to do in quantum mechanics. We start with the Lagrangian density of the system. The Klein-Gordon equation gives the dynamics of a free scalar field. According to [5], the Lagrangian and Hamiltonian whose equations of motion describe this dynamics are

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2, \quad (2.7)$$

and

$$\mathcal{H} = \frac{1}{2} \Pi^2 + \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{m^2}{2} \phi^2, \quad (2.8)$$

where  $\Pi$  is the conjugated momentum. In the canonical quantization scheme, we impose the commutation relation between the field and its conjugated momentum in equal times as being

$$[\phi(x), \Pi(y)]_{x_0=y_0} = i\delta^3(x - y). \quad (2.9)$$

The solution for equation (2.2) then is given by

$$\phi(x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2p^0}} (e^{-ip \cdot x} \phi(\vec{p}) + e^{ip \cdot x} \phi^\dagger(\vec{p})), \quad (2.10)$$

and its conjugated momentum

$$\Pi(x) = \frac{\partial \phi(x)}{\partial x_0} = -i \int d^3 p \sqrt{\frac{p^0}{2(2\pi)^3}} (e^{-ip \cdot x} \phi(\vec{p}) - e^{ip \cdot x} \phi^\dagger(\vec{p})), \quad (2.11)$$

where the integrals are taken over the momentum spatial components. The factor  $p^0$  is the zero component of the momentum. The operators  $\phi^\dagger(\vec{p})$  and  $\phi(\vec{p})$  are called the creation and annihilation operators respectively. Equations (2.10) and (2.11) lead to the commutation relation between the Fourier expansion modes

$$[\phi(\vec{p}), \phi^\dagger(\vec{q})] = \delta^3(p - q). \quad (2.12)$$

We define the ground state of the theory as being annihilated by the annihila-

tion operator, that is

$$\phi(\vec{p}) |0\rangle \equiv 0. \quad (2.13)$$

The creation operator  $\phi^\dagger(\vec{p})$  creates particles on the ket with momentum  $p$ , i.e.

$$\phi^\dagger(\vec{p}) |0\rangle = |p\rangle, \quad (2.14)$$

where  $|p\rangle$  is a single particle state with momentum  $p$ . Let us consider  $n$  Hilbert spaces  $\mathcal{H}$  such that

$$\begin{aligned} |0\rangle &\rightarrow \mathcal{H}_0 \\ \phi^\dagger(p) |0\rangle &= |p\rangle \rightarrow \mathcal{H}_1 \\ \phi^\dagger(p_1)\phi^\dagger(p_2) |0\rangle &= |p_1, p_2\rangle \rightarrow \mathcal{H}_2. \end{aligned} \quad (2.15)$$

The first is a Hilbert space of zero particles, the second is the space of one particle, and so on. The sum of all these different Hilbert spaces for all the possible  $n$  particles generates what is called the Fock space, and this is where QFT is defined. This allows the particle-antiparticle pair creation, since we are now able to transit between different Hilbert spaces.

Besides the field operators being a solution to our previous problems, the Hamiltonian given by

$$H = \int d^3x \mathcal{H} = \int d^3x \left( \frac{1}{2} \Pi^2 + \frac{1}{2} \nabla\phi \cdot \nabla\phi + \frac{m^2}{2} \phi^2 \right) \quad (2.16)$$

also has a positive semi definite spectrum [5], which solves the Klein-Gordon paradox. We now consider the solution for the Dirac Equation.

#### 2.1.4 Dirac Field Theory

Different from the Klein-Gordon equation, the Dirac equation is applied to objects with a column vector structure, which suggests that the solutions should describe particles with spin [21]. Indeed, the Dirac equation is applied to spinors with  $s = \frac{1}{2}$  [5]. Another main difference, as we are dealing with half-integer spin particles, is that we should use the Fermi-Dirac statistics, using anti-commutation rules.

The Lagrangian density and the Hamiltonian density that correspond to the dynamics of the Dirac equation, without interaction, are given by

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi, \quad (2.17)$$

and

$$\mathcal{H} = -i\bar{\psi}\boldsymbol{\gamma} \cdot \nabla\psi + m\bar{\psi}\psi, \quad (2.18)$$

where  $\bar{\psi} \equiv \psi^\dagger\gamma^0$ . The field operators  $\psi$  and  $\psi^\dagger$  must be treated as two independent field operators, as each satisfy their respective Dirac equation. We want to quantize the fields under the canonical scheme, so the anti-commutation relations between the spinor components  $\psi_\alpha(x)$  and  $\psi_\beta(y)$  at equal times are imposed to be

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\}_{x_0=y_0} = \delta_{\alpha\beta}\delta^{(3)}(x-y), \quad (2.19)$$

with other anti-commutation relations vanishing.

The field solutions are

$$\psi_\alpha(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} \left( e^{-ik\cdot x} a(\vec{k}, s) u_\alpha(k, s) + e^{ik\cdot x} b^\dagger(\vec{k}, s) v_\alpha(k, s) \right), \quad (2.20)$$

and

$$\psi_\alpha^\dagger(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} \left( e^{-ik\cdot x} b(\vec{k}, s) u_\alpha^*(k, s) + e^{ik\cdot x} a^\dagger(\vec{k}, s) v_\alpha^*(k, s) \right). \quad (2.21)$$

The integrals are taken over the spacial components and  $k^0$  is the energy of the 4-component momentum. The sum is over the possible values of spin;  $u_\alpha(k, s)$  and  $v_\alpha(k, s)$  are spinors that satisfy the positive and the negative energy Dirac equations respectively. They define a complete basis for the spinors;  $\alpha$  represents the spinor component; and  $(a^\dagger, b^\dagger)$  and  $(a, b)$  are the creation and annihilation operators respectively. These solutions imply in the anti-commutator rule for the creation and annihilation operators as

$$\{a(\vec{k}, s), a^\dagger(\vec{k}', s')\} = \{b(\vec{k}, s), b^\dagger(\vec{k}', s')\} = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \delta_{ss'}. \quad (2.22)$$

The factor  $(2\pi)^3$  and the fact that the Dirac-delta has three components is due the fact we are defining our operator for three space dimensions.

The operator  $a^\dagger$  creates a particle with positive energy  $E_p = \sqrt{p^2 - m^2} > 0$  and positive charge, while  $b^\dagger$  creates an anti-particle with the same positive energy and negative charge [1], that is,

$$a^\dagger(\vec{p}) |0\rangle = |p\rangle \quad (2.23)$$

$$b^\dagger(\vec{p}) |0\rangle = |\tilde{p}\rangle. \quad (2.24)$$

Also, the annihilation operators are defined so that

$$a(\vec{p}) |0\rangle = 0 \quad (2.25)$$

$$b(\vec{p}) |0\rangle = 0. \quad (2.26)$$

The Hamiltonian of the theory is given by

$$H = \int d^3x \mathcal{H} = \int d^3x -i\bar{\psi}\gamma \cdot \nabla\psi + m\bar{\psi}\psi. \quad (2.27)$$

When we quantize the theory with anti-commutation rules and anti-particle operators, the Hamiltonian becomes positive defined and the system is bounded from below, that is, the vacuum is the state with no particles or anti-particles and it has zero energy, the lowest energy of the system. Now that we have the field solutions for fermions (spinor fields) and bosons (scalar fields), we can analyze how to compute correlation functions of a system, features which are really important in QFT.

### 2.1.5 Path Integral and the Feynman Green Function

The full development of the path integral can be found in Chapters 2 and 4 from reference [21]. Here we use the path integral to describe a field theory and use this description to compute, for example, Green functions. An object of great interest in QFT is the vacuum functional, which can be used to generate the Green functions of the theory. If one has these functions, one may construct the S-matrix and therefore solve scattering processes, which may be used to study the theory. In this subsection we develop the calculations for the scalar field theory. The same process can be used for the fermionic case. The vacuum functional is defined as

$$Z[J] \equiv N \int D\phi e^{iS[\phi, J]}, \quad (2.28)$$

where  $D\phi$  is the integration measure of the path integral and  $N$  is a normalization constant normally chosen so that  $Z[0] = 1$ .  $J$  is an external source added to the system with the solo purpose to allow some calculations to be performed, and it is usually set to zero at the end.  $S[\phi, J]$  is the action of the system plus a source term, defined as

$$S[\phi, J] \equiv S[\phi] + \int d^4x J(x)\phi(x). \quad (2.29)$$

For the Klein-Gordon equation, the Green function is a function  $G(x - x')$  that satisfies

$$(\square + m^2) G(x - x') = -\delta^4(x - x'). \quad (2.30)$$

This function can be found by going to the momenta space, that is

$$G(x - x') = \int \frac{d^4x}{(2\pi)^2} G(k) e^{-ik \cdot (x-x')}, \quad (2.31)$$

$$\delta^4(x - x') = \int \frac{d^4x}{(2\pi)^4} e^{-ik \cdot (x-x')}. \quad (2.32)$$

If we replace these two equations into equation (2.30), we get that

$$G(k) = \frac{1}{(2\pi)^2} \frac{1}{k^2 - m^2}, \quad (2.33)$$

and

$$G(x - x') = \int \frac{d^4x}{(2\pi)^4} \frac{1}{k^2 - m^2} e^{-ik \cdot (x-x')}. \quad (2.34)$$

The Feynman's Green function ( $G_F$ ) is defined to be the time ordered Green function that satisfies the equations of motion of the system [12], and for the Klein-Gordon equation is defined as

$$G_F(x - x') \equiv \lim_{\epsilon \rightarrow 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x-x')}. \quad (2.35)$$

The  $+i\epsilon$  factor guarantees that our function is well defined for all values of  $k^2$ . The Feynman's Green function is widely used in QFT because it leads us to time ordered quantities [1], and it also satisfies equation (2.30).  $G_F$  is also called the propagator of the theory.

In quantum field theory we often want to evaluate the expected value of an operator  $\mathcal{O}$  in order to find physical quantities. This value is defined to be

$$\langle 0|T\mathcal{O}|0\rangle \equiv N \int D\phi \mathcal{O} e^{iS[\phi, J]} \Big|_{J=0}. \quad (2.36)$$

In the above expression the right hand side is calculated in the  $i\epsilon$  prescription and  $T$  means time ordered operator. Particularly there is one function of great interest, the time ordered two point function given by  $\mathcal{O} = \phi(x)\phi(x')$ . We want to see how this function is connected to Feynman's propagator.

In the presence of a source, the time ordered two point function is

$$\begin{aligned} \langle 0|T(\phi(x)\phi(x'))|0\rangle &= N \int D\phi \phi(x)\phi(x') e^{iS[\phi, J]} \\ &= N \lim_{\epsilon \rightarrow 0^+} \int D\phi \phi(x)\phi(x') e^{-i \int d^4x \left[ \frac{1}{2} \phi(x)(\partial_\mu \partial^\mu - \frac{m^2}{2} - i\epsilon)\phi(x) - J(x)\phi(x) \right]} \\ &= N \lim_{\epsilon \rightarrow 0^+} \int D\phi \frac{-\delta}{\delta J(x)} \frac{\delta}{\delta J(x')} e^{-i \int d^4x \left[ \frac{1}{2} \phi(x)(\partial_\mu \partial^\mu - \frac{m^2}{2} - i\epsilon)\phi(x) - J(x)\phi(x) \right]}. \end{aligned} \quad (2.37)$$

We now perform a variable change

$$\tilde{\phi}(x) = \phi(x) + \int d^4z G_F(x-z)J(z), \quad (2.38)$$

with  $G_F$  defined in equation (2.35). Replacing this variable change in equation (2.37) we have

$$\begin{aligned} \langle 0|T(\phi(x)\phi(x'))|0\rangle &= N \lim_{\epsilon \rightarrow 0^+} \int D\tilde{\phi} \frac{\delta}{\delta J(x)} \frac{-\delta}{\delta J(x')} e^{-\frac{i}{2} \int \int d^4z d^4z' J(z)G_F(z-z')J(z')} e^{-i \int d^4x \frac{1}{2}\tilde{\phi}(x)(\partial_\mu\partial^\mu - \frac{m^2}{2} - i\epsilon)\tilde{\phi}(x)} \\ &= N[\det(\partial_\mu\partial^\mu + m^2)]^{-\frac{1}{2}} \frac{-\delta}{\delta J(x)} \frac{\delta}{\delta J(x')} e^{-\frac{i}{2} \int \int d^4z d^4z' J(z)G_F(z-z')J(z')} \\ &= Z[0]iG_F(x-x')e^{-\frac{i}{2} \int \int d^4z d^4z' J(z)G_F(z-z')J(z')} + \\ &+ Z[0] \int \int d^4z d^4z' G_F(x-z)G_F(x'-z')J(z)J(z')e^{-\frac{i}{2} \int \int d^4z d^4z' J(z)G_F(z-z')J(z')}. \end{aligned} \quad (2.39)$$

Considering  $Z[0] = 1$  and taking the source term  $J$  to zero, we obtain that

$$\langle 0|T(\phi(x)\phi(x'))|0\rangle = iG_F(x-x'). \quad (2.40)$$

Lastly, if the action of the system has an interaction term, that is

$$S = S[\phi, J] + S_{int}, \quad (2.41)$$

the two point function of the system becomes

$$\begin{aligned} \langle 0|T\phi(x)\phi(x')|0\rangle_{int} &= N \int D\phi \phi(x)\phi(x')e^{iS} \\ &= N \int D\phi \phi(x)\phi(x')e^{i \int S[\phi, J] + iS_{int}} \\ &= \langle 0|T\phi(x)\phi(x')e^{iS_{int}}|0\rangle, \end{aligned} \quad (2.42)$$

such that the the expected value with the index *int* means that we are evaluating the propagator in the interacting theory, with the action described by equation (2.41), and the absence of the index means that we are considering only the action given by equation (2.29). Having the basic tools for quantum field theory, we move forward and start working with the Renormalization Group.

## 2.2 RENORMALIZATION GROUP

This section has the purpose of briefly explaining what is the Renormalization Group (RG) and how to use it. The modern concepts of the Renormalization Group was first introduced by Kadanoff [22] and Wilson [23], with the purpose to define a Quantum Fi-

old Theory outside the framework of perturbation theory. The easiest way to define it is as a transformation, mapping a system in a energy scale (at short distances or high-energy cutoff) into another equivalent system in a different scale [9]. We may use the RG in QFT to evaluate whether interactions in a system are relevant or not when we analyze this system in different energy scales.

There are two steps that one may follow to apply the RG: Integrating out high energy scales and rescaling lengths [9]. Basically, some degrees of freedom representing the high energy physics are integrated out, followed then by a scale transformation to restore the original units. Let us see how to implement these steps.

### 2.2.1 Integrating out Fast Modes and Rescaling

First, we consider the path integral approach to a field system in  $D$  space-time dimensions.

$$Z = \int D\phi e^{-S(\phi)}, \quad (2.43)$$

being  $S(\phi)$  the action of the system and  $Z$  the partition function. The system can obey Fermi or Bose statistics. The field  $\phi(x)$  can be expanded in Fourier modes as

$$\phi(x) = \int \frac{d^D k}{(2\pi)^D} e^{ikx} \phi(k). \quad (2.44)$$

This equation is valid if we consider that this system has a high energy cutoff  $\Lambda$ , being  $|k| < \Lambda$  [13].

Our goal is to integrate out some degrees of freedom representing the fast modes (high momenta), and define an effective action  $S_{eff}$ . We define  $\phi_f(x)$  to be the fast modes, with high energies and short distances scale, and  $\phi_s(x)$  to be the slow modes, with smaller energies and long distances scale. Then we can write  $\phi(x)$  to be

$$\phi = \phi_s + \phi_f. \quad (2.45)$$

Replacing it into equation (2.44),

$$Z = \int D\phi_s D\phi_f e^{-S(\phi_s + \phi_f)} \equiv \int D\phi_s e^{-S_{eff}(\phi_s)}. \quad (2.46)$$

This procedure is only a formal structure. It is not always simple to integrate the fast modes. The result given by equation (2.46) has a smaller energy cutoff  $\Lambda' = b\Lambda$  ( $b < 1$ ), since we integrated out the fast modes [13]. This result is just fine. We could have stopped there, but we wouldn't be able to compare equation (2.46) to our earlier action in (2.43). We have to be at the same units or scale. In order to solve that problem we need to rescale lengths, time

and momenta as

$$x = \frac{x'}{b}, \quad (2.47)$$

$$k = bk', \quad (2.48)$$

such that  $|k'| < \Lambda$  again.

The form of the effective action  $S_{eff}$  is clearly affected by the rescale of the variables, which can be implemented in several ways. However, they must preserve the symmetries of the system [3]. If we have an action  $S^*$  that remains invariant under the RG transformation, that is,

$$S_{eff}^*(\phi_s) = S^*(\phi), \quad (2.49)$$

we call this action a fixed point of the RG group, and it can be stable or unstable. At a fixed point, when we apply the RG transformation to the Hamiltonian or the action of the system, we get the same fixed points. Thus, the system at a fixed point has a scale invariance symmetry [13]. This means that can be no scale left in the system, meaning that this point describes two types of systems: a system with vanishing correlation length and divergent energy gap, or a system with divergent correlation length and vanishing energy gap. For the first case, the point is called a stable fixed point, because it remains stable under local perturbations. The second is called an unstable fixed point, hence they are unstable to at least one local perturbation (or more) [9]. We can now analyze how a perturbation acts on a system that is at a fixed point.

### 2.2.2 Relevant, Irrelevant and Marginal operators

Consider a system close to a fixed point. The action can be represented by a scale invariant part plus a set of operators. Explicitly,

$$S(\phi) = S^*(\phi) + \int d^D x \sum_n \lambda_n \phi_n(x), \quad (2.50)$$

being  $\phi_n(x)$  a set of  $n$  operators and  $\lambda_n$  their respective coupling constants. The first part  $S^*$  is invariant under the RG transformation<sup>1</sup>, but we may now see how the operators transform as well. Under the rescale  $x \rightarrow b^{-1}x'$ . By dropping primes we get

$$\phi_n(b^{-1}x) = b^{\Delta_n} \phi_n(x), \quad (2.51)$$

---

<sup>1</sup>We are considering the first part  $S^*$  to be invariant under the RG transformation, which may not be the case for all fixed points since it now depends on the scaling dimension of the operator  $\phi_n(x)$ . In this work, we will analyze actions that satisfy this requirement at a *tree*-level of the Beta function.

where  $\Delta_n$  is the scaling dimension of the operator. If we replace this result into the integral of equation (2.50), remembering that we must rescale the integrating factor as well, we get

$$\int d^D x \sum_n \lambda_n \phi_n(x) = \int d^D x \sum_n b^{-D+\Delta_n} \lambda_n \phi_n(x). \quad (2.52)$$

In that sense, we could replace the rescale of the field with a rescale of the coupling constant  $\lambda_n$  given by

$$\lambda'_n = \lambda_n b^{\Delta_n - D} \equiv \lambda_n(b). \quad (2.53)$$

Since  $b < 1$  we can parameterize to be  $b = e^{-\delta\mu}$  with  $\mu$  being a small dimensionless energy-scale parameter, such that we may transform the equation for the coupling constant in a differential equation, known as the beta function.

$$\beta(\lambda_n) = \frac{\partial \lambda_n}{\partial \mu} = (D - \Delta_n) \lambda_n + \dots \quad (2.54)$$

Let us see what equation (2.54) means. First, we are considering only the first non-vanish contribution at a *tree*-level, since higher order contributions may not let  $S^*$  invariant. As we are interested in long distance scale physics, the coupling constants can get smaller or bigger. If they tend to get smaller each time we perform a RG transformation, we will get to the point where we can neglect this term, meaning that the operator related to this term is irrelevant. On the other hand, if the constant tends to get bigger, this term will become more relevant as we perform the transformations, so the operator related is called relevant. Given  $b < 1$ , there is a simple rule to define if either the operators corresponding to these coupling constants are relevant, irrelevant or marginal: (i) If  $(D - \Delta_n) > 0$ , the coupling constant tends to grow as we go to a smaller momentum scale, and the operator  $\phi_n$  is called relevant. This type of operator tends to drive the action away from the fixed point described by  $S^*$ , into another fixed point. (ii) If  $(D - \Delta_n) < 0$ , the coupling constant decreases as we go to a smaller momentum scale, and the operator  $\phi_n$  is called irrelevant. This operator can be neglected at a low-energy limit. (iii) The last case is when  $(D - \Delta_n) = 0$  to leading order. The operator  $\phi_n$  is then called a marginal operator. We need to perform further calculations to see how this operator evolve under the RG transformation.

The concept of the renormalization group will become clearer when we apply it to a system, which will be done in future chapters of this work, but this section may be enough for refreshing the memory. Now that we reviewed the basic tools needed for most calculations of this thesis, we can start looking at more complicated topics. The first one is bosonization.

### 3 BOSONIZATION

In this chapter, we are going to study bosonization. This tool refers to the possibility of describing a theory of Dirac fermions by a bosonic field theory described by integer spin particles called bosons. This has been mostly used in 1+1 dimensions, where there is a map between the fermionic operator and the bosonic operator [3]. One of the pioneers of this technique was Haldane, by the study of Luttinger Liquid in one dimension [24]. By the exchange of a fermion problem for a bosonic one, some calculations can become quite simpler, as we will see in the latter sections. First, we need to study the fermionic field system and the bosonic field system in 1+1 dimensions, with the goal to construct the bosonization dictionary. If we have a complete understanding about these systems, we will be able to check if the bosonization fully maps one fermion system into a boson one, and vice-versa.

In the following sections, we follow the calculations from references [4, 3, 5]. We are using the same unit system and notation defined in the previous chapter, with the slight change that  $x$  represents the spatial component and  $t$  the temporal dimension. For 1 + 1 dimensions, we use the metric  $(+, -)$ .

#### 3.1 MASSLESS DIRAC FERMION FIELD

We will study a massless system composed by Dirac fermions in 1+1 dimensions. This structure describes particles with spin  $\frac{1}{2}$ , such as the electron, respecting the Fermi-Dirac statistics, where each energy state can be occupied by one or zero particles. We use anti-commutation rules to quantize. The Lagrangian and Hamiltonian densities in 1+1 dimensions for massless Dirac fermions are given by

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi, \quad (3.1)$$

and

$$\mathcal{H} = -i \bar{\psi} \gamma^1 \partial_x \psi. \quad (3.2)$$

In 1+1 dimensions, we just need the gamma matrices to be two square matrices of order 2 that respect the Clifford algebra, that is,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}. \quad (3.3)$$

With that in mind, we choose them to be

$$\gamma^0 \equiv \sigma_1, \quad (3.4)$$

$$\gamma^1 \equiv i\sigma_2, \quad (3.5)$$

which implies that

$$\gamma^0\gamma^1 = \gamma^5 \equiv \sigma_3, \quad (3.6)$$

where  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  are the Pauli matrices. We are also going to use the momentum operator  $P = -i\partial_x$ . Replacing these quantities into the Hamiltonian density, the Hamiltonian of the system becomes

$$H = \int_{-\infty}^{\infty} dx \psi^\dagger(x)(\sigma_3 P)\psi(x). \quad (3.7)$$

The field  $\psi$  is a two component spinor in 1+1 dimensions. We can divide the components in  $\psi_+$  and  $\psi_-$ , resulting in

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \quad (3.8)$$

In order to analyze the dynamics of these components, we go back to the Dirac equation in (2.5). Replacing the fields by its components from equation (3.8), we get

$$\begin{bmatrix} (\partial_t + \partial_x)\psi_- \\ (\partial_t - \partial_x)\psi_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (3.9)$$

We can see that with an increase of time, the spatial coordinate of  $\psi_-$  tends to decrease, which means that this component represents a fermion moving to the left. For  $\psi_+$ , an increase of time implies an increase in the spatial coordinate, representing a fermion moving to the right. For that reason,  $\psi_-$  is called a left mover and  $\psi_+$  is called a right mover.

In the second quantization, the field operators  $\psi_+$  and its hermitian conjugate  $\psi_+^\dagger$  are taken to be independent operators. They have non-trivial anti-commutation rules [5], given by

$$\{\psi_\pm^\dagger(x), \psi_\pm(y)\} = \delta(x - y), \quad (3.10)$$

with other anti-commutation rules defined as zero.

Now we want to go to the momentum space. Thus, the Fourier transform, the inverse transform and the new anti-commutation rules are respectively

$$\psi_\pm(p) = \int_{-\infty}^{\infty} dx \psi_\pm(x)e^{-ipx}, \quad (3.11)$$

$$\psi_{\pm}(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \psi_{\pm}(p) e^{ipx} e^{-\frac{\alpha}{2}|p|}, \quad (3.12)$$

$$\{\psi_{\pm}^{\dagger}(p), \psi_{\pm}(q)\} = 2\pi\delta(p - q), \quad (3.13)$$

where  $\alpha$  is a convergence factor set to 0 at the end, since in QFT we usually have ultraviolet divergences and we may have some problems when computing correlation functions for operators separated by a distance  $x \rightarrow 0$ . Instead of inserting a cutoff at the integral limits, which would make a lot harder for us to perform some calculations, we insert this convergence factor to force the functions to be smooth at this divergent limit. The field operators  $\psi^{\dagger}(p)$  and  $\psi(p)$  are the creation and annihilation operators.

Using the notation from equation (3.8), we can rewrite the Hamiltonian in the momentum space as

$$\begin{aligned} H &= \int dx \psi^{\dagger}(x) (\sigma_3 P) \psi(x) \\ &= \int dx \psi_+^{\dagger}(x) (-i\partial_x) \psi_+(x) + \psi_-^{\dagger}(x) (i\partial_x) \psi_-(x) \\ &= \int \frac{dp}{2\pi} \frac{dq}{2\pi} dx \psi_+^{\dagger}(p) (-i\partial_x) e^{ix(p-q)} \psi_+(q) e^{-\frac{\alpha}{2}(|p|+|q|)} \\ &\quad + \int \frac{dp}{2\pi} \frac{dq}{2\pi} dx \psi_-^{\dagger}(p) (i\partial_x) e^{ix(p-q)} \psi_-(q) e^{-\frac{\alpha}{2}(|p|+|q|)} \\ &= \int \frac{dp}{2\pi} \psi_+^{\dagger}(p) p \psi_+(p) - \psi_-^{\dagger}(p) p \psi_-(p), \end{aligned} \quad (3.14)$$

where we used the fact that  $\psi_+$  anti-commute with  $\psi_-$ ,  $P = -i\partial_x$  and  $\alpha = 0$ . If we analyze equation (3.14), we can observe the spectrum of the right and left movers in the momentum space, given by figure 3.1.

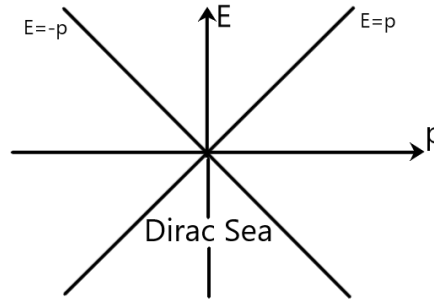


Figure 3.1: The spectrum of the right movers, represented by the line  $E = p$ , and the spectrum for left movers, represented by the line  $E = -p$

From the figure above, if we consider the spectrum for all values of  $p$ , we obtain negative energy values for right movers with negative momenta and for left movers with positive momenta. To avoid this problem, we consider the existence of the Dirac sea, where

all the negative energy states are filled. In standard quantum field theory, we define our system with anti-particle and particle operators so our vacuum  $|0\rangle$  is the state containing no particles or anti-particles. In the Dirac sea interpretation, this state is obtained when we consider all the negative energy states filled, that is,

$$|0\rangle \equiv \prod_{p<0} \prod_{q>0} \psi_{-}^{\dagger}(q) \psi_{+}^{\dagger}(p) |0_D\rangle, \quad (3.15)$$

where the Dirac vacuum  $|0_D\rangle$  is now the state with no particles. The way we defined our operators, this interpretation is necessary so our system properly describes a quantum field theory.

Before computing the correlation functions of the theory, we can see how the field operators evolve in the Heisenberg picture. Performing a Fourier expansion in the time dependent operator, we have

$$\psi_{\pm}(x, t) = \int_{-\infty}^{\infty} \frac{dp dp_0}{2\pi} \psi_{\pm}(p_0, p) e^{-ip_0 t + ipx} e^{\frac{-\alpha}{2}|p|}. \quad (3.16)$$

If we apply the relation in equation (3.9), the only non-vanishing contributions of the Fourier transform are

$$\begin{aligned} \psi_{\pm}(x, t) &= \int_{-\infty}^{\infty} \frac{dp dp_0}{2\pi} \psi_{\pm}(p_0, p) \delta(p \pm p_0) e^{-ip_0 t + ipx} e^{\frac{-\alpha}{2}|p|} \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \psi_{\pm}(p) e^{ip(x \mp t)} e^{\frac{-\alpha}{2}|p|}, \end{aligned} \quad (3.17)$$

where in the case of  $\psi_{+}$ , we had to make a variable change of  $p_0 = -p_0$  in the last line. We can see that  $\psi_{\pm}$  are functions of  $t \mp x$ , which once again shows that  $\psi_{+}$  represents fermions moving to the right and  $\psi_{-}$  represents fermions moving to the left. We define the anti-commutation relation and the correlation functions at equal times, and so the time exponential in equation (3.17) will have no contribution in our calculations.

Having defined the field operators and the Hamiltonian, we now want to focus on the correlation functions of the system. They will play a main role in the bosonization dictionary by the fact that they will be used to prove the fermion-boson correspondence [4]. Let us see some correlation functions that are non-trivial. For example

$$\langle \psi_{+}(x) \psi_{+}^{\dagger}(0) \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{dq}{2\pi} e^{-\frac{1}{2}\alpha(|p|+|q|)} e^{ipx} \langle \psi_{+}(p) | \psi_{+}^{\dagger}(q) \rangle \quad (3.18)$$

$$\psi_{+}^{\dagger}(p') |0\rangle = \prod_{p<0} \prod_{q>0} \psi_{+}^{\dagger}(p') \psi_{-}^{\dagger}(q) \psi_{+}^{\dagger}(p) |0_D\rangle. \quad (3.19)$$

Because of the Pauli exclusion principle, if  $p' = p$ , we will have  $(\psi_{+}^{\dagger}(p))^2 = 0$ . This means that

the only non-vanishing contributions is when  $p' > 0$ , because  $p$  only assumes negative values. The practical effect is that the right movers only can create fermions with positive momenta above the Dirac sea [25]. In order to restrict the expected value only for positive momenta in equation (3.20), we insert a step function, so the equation becomes

$$\begin{aligned}
\langle \psi_+(x) \psi_+^\dagger(0) \rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{dq}{2\pi} e^{-\frac{1}{2}\alpha(|p|+|q|)} e^{ipx} \Theta(q) \langle p|q \rangle \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{dq}{2\pi} e^{-\frac{1}{2}\alpha(|p|+|q|)} e^{ipx} \Theta(q) 2\pi \delta(p-q) \\
&= \int_0^{\infty} \frac{dp}{2\pi} e^{-\alpha|p|} e^{ipx} \\
&= \frac{1}{2\pi} \frac{1}{\alpha - ix}.
\end{aligned} \tag{3.20}$$

The step function represented by  $\Theta(q)$  guarantees that we only consider the non-vanishing values of the right movers acting on the bracket. There is no need in adding another step function for  $\Theta(p)$ , as the delta function will make  $p = q$ . We can also use the same construction of equation (3.19) to see that the left movers can only create particles with negative momenta above the Dirac sea.

Following the same path of equation (3.20), for the left movers we have

$$\begin{aligned}
\langle \psi_-(x) \psi_-^\dagger(0) \rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{dq}{2\pi} e^{-\frac{1}{2}\alpha(|p|+|q|)} e^{ipx} \langle \psi_-(p) \psi_-^\dagger(q) \rangle \\
&= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{dq}{2\pi} e^{-\frac{1}{2}\alpha(|p|+|q|)} e^{ipx} \Theta(-q) 2\pi \delta(p-q) \\
&= \int_{-\infty}^0 \frac{dp}{2\pi} e^{-\alpha|p|} e^{ipx} \\
&= \frac{1}{2\pi} \frac{1}{-\alpha - ix},
\end{aligned} \tag{3.21}$$

where now the theta function must be negative so the expected value does not vanish. Also, note that

$$\langle \psi_\pm^\dagger(0) \psi_\pm(x) \rangle = \frac{\mp i}{\mp i\alpha + x}. \tag{3.22}$$

Lastly, we need to analyze the currents related to the fermion field. Under a  $U(1)$  transformation, that is,  $\psi' \rightarrow e^{-i\alpha}\psi$  and  $\psi'^\dagger \rightarrow e^{i\alpha}\psi^\dagger$  for an small constant value of  $\alpha$ , the Lagrangian in equation (3.1) remains invariant. We can compute a conserved current  $j^\mu$  related to the transformation as

$$j_\alpha^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \delta_\alpha \psi^\dagger + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta_\alpha \psi. \tag{3.23}$$

Under this transformation,

$$\delta_\alpha \psi = \psi' - \psi = -i\alpha\psi, \quad (3.24)$$

and

$$\delta_\alpha \psi^\dagger = \psi^{\dagger'} - \psi^\dagger = i\alpha\psi^\dagger. \quad (3.25)$$

Replacing these results in equation (3.23), we have

$$j_\alpha^\mu = \alpha \bar{\psi} \gamma^\mu \psi = \alpha j^\mu, \quad (3.26)$$

where the conserved current is

$$j^\mu = \bar{\psi} \gamma^\mu \psi, \quad (3.27)$$

with components

$$\begin{aligned} j_0 &= \psi^\dagger \gamma^0 \gamma^0 \psi = \psi^\dagger \psi \\ &= \psi_+^\dagger(x) \psi_+(x) + \psi_-^\dagger(x) \psi_-(x); \end{aligned} \quad (3.28)$$

$$\begin{aligned} j_1 &= \psi^\dagger \sigma_3 \psi \\ &= \psi_+^\dagger(x) \psi_+(x) - \psi_-^\dagger(x) \psi_-(x). \end{aligned} \quad (3.29)$$

The component  $j_0$  is related to charge conservation and  $j_1$  is related to a translational symmetry. This section was devoted to review some concepts and calculations about the fermion field. The next step is to do a similar study about a boson field, so that we can compare both systems later on.

### 3.2 FREE MASSLESS SCALAR FIELD

Scalar fields describe integer spin particles called bosons. This type of particles respect the Bose-Einstein statistics, where more than one particle can exist at the same state [5]. Here, we quantize the system with commutation rules. We shall start with Lagrangian of the massless system, in 1+1 dimensions, given by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi. \quad (3.30)$$

We can also compute the Hamiltonian of the system, such that

$$H = \frac{1}{2} \int dx (\Pi^2 + (\partial\phi)^2), \quad (3.31)$$

where  $\phi$  is the scalar field operator and  $\Pi$  is its conjugated momentum

$$\begin{aligned} \Pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ &= \dot{\phi}. \end{aligned} \quad (3.32)$$

In the canonical scheme, the operators respect the commutation rule

$$[\phi(x), \Pi(y)] = i\delta(x - y). \quad (3.33)$$

The operator in the Schrödinger picture that solves the motion equations of our system is

$$\phi(x) = \int \frac{dp}{2\pi\sqrt{2|p|}} [\phi(p)e^{ipx} + \phi^\dagger(p)e^{-ipx}] e^{-\frac{\alpha}{2}|p|}, \quad (3.34)$$

being  $\alpha$  a convergence factor set to 0 at the end of the calculation just as we did for the fermionic case. To calculate  $\Pi$ , we just need the time derivative of the field operator. Just as in the fermionic case, we can find the time dependent operator in the Heisenberg picture expanding the operator in Fourier modes for the time and spatial coordinate, and use the Klein-Gordon equation in (3.33) to select only the non-vanishing values. Following this path, the time dependent operator is

$$\phi(x, t) = \int \frac{dp}{2\pi\sqrt{2|p|}} [\phi(p)e^{ipx - i|p|t} + \phi^\dagger(p)e^{-ipx + i|p|t}] e^{-\frac{\alpha}{2}|p|}, \quad (3.35)$$

and so we can compute the conjugated momentum in the Schrödinger picture as

$$\Pi(x) = \int \frac{dp|p|}{2\pi\sqrt{2|p|}} [-i\phi(p)e^{ipx} + i\phi^\dagger(p)e^{-ipx}] e^{-\frac{\alpha}{2}|p|}. \quad (3.36)$$

Having the explicit forms of  $\phi$  and  $\Pi$ , we can replace equation (3.34) and (3.36) in equation (3.33) to see that  $\phi^\dagger(p)$  and  $\phi(p)$  are the creation and annihilation operator of this theory respectively, as the non-trivial commutation rule between these operators is

$$[\phi(p), \phi^\dagger(q)] = 2\pi\delta(p - q). \quad (3.37)$$

This relation arises naturally to maintain our previous commutation relations in the canonical scheme. Knowing that these operators are the creation and annihilation operators, they act on

the vacuum state as

$$\phi^\dagger(p) |0\rangle \equiv |p\rangle \quad (3.38)$$

and

$$\phi(p) |0\rangle \equiv 0. \quad (3.39)$$

Applying equations (3.34) and (3.36) in (3.31) to the Hamiltonian, we have

$$H = \int \frac{dp}{2\pi} |p| \phi^\dagger(p) \phi(p). \quad (3.40)$$

Now, we can see that the spectrum of the system is  $E = |p|$ , since the Hamiltonian is in its diagonal form.

We follow the same path we did for fermions. We will introduce now right and left movers in the coordinate-space  $\phi_\pm(x)$ . Being  $\phi$  a scalar field, we define

$$\phi(x) \equiv \phi_+(x) + \phi_-(x), \quad (3.41)$$

where, in the Schrödinger picture,

$$\phi_\pm(x) = \pm \int_0^{\pm\infty} \frac{dp}{2\pi\sqrt{2}|p|} e^{-\frac{1}{2}\alpha|p|} [\phi(p)e^{ipx} + \phi^\dagger(p)e^{-ipx}]. \quad (3.42)$$

Also, comparing this operator with equation (3.35) in the Heisenberg picture, the time dependent operators must be

$$\phi_\pm(x, t) = \pm \int_0^{\pm\infty} \frac{dp}{2\pi\sqrt{2}|p|} e^{-\frac{1}{2}\alpha|p|} [\phi(p)e^{ip(x\mp t)} + \phi^\dagger(p)e^{-ip(x\mp t)}]. \quad (3.43)$$

This last equation shows us that  $\phi_\pm$  are functions of  $x \mp t$ , which shows that  $\phi_+$  represents right-moving bosons and  $\phi_-$  represents left-moving bosons. Alike the fermionic case, we will work in the Schrödinger picture.

In order to compute the commutation rules between the operators, we rewrite the movers as

$$\phi_\pm(x) = \frac{1}{2} \left[ \phi(x) \mp \int_{-\infty}^x \Pi(x') dx' \right], \quad (3.44)$$

such that the commutation relations are

$$\begin{aligned}
[\phi_{\pm}(x), \phi_{\pm}(y)] &= \mp \frac{1}{4} \int_{-\infty}^y dy' [\phi(x), \Pi(y')] \mp \frac{1}{4} \int_{-\infty}^x dx' [\Pi(y), \phi(x')] \\
&= \mp \frac{1}{4} \int_{-\infty}^y dy' i\delta(x - y') \pm \frac{1}{4} \int_{-\infty}^x dx' i\delta(x' - y) \\
&= \pm \frac{i}{4} \text{sign}(x - y).
\end{aligned} \tag{3.45}$$

In the last passage, if  $x < y$ , the first integral is non-vanishing because the Dirac delta is in the integration interval while the second delta is not, forcing the second integral to be zero. For  $x > y$ , the first integral vanishes and the second has a finite value. These two conditions result in the commutation rule given by equation (3.45). We can also compute the commutation between right and left movers, given by

$$\begin{aligned}
[\phi_+(x), \phi_-(y)] &= \frac{1}{4} \int_{-\infty}^y dy' [\phi(x), \Pi(y')] - \frac{1}{4} \int_{-\infty}^x dx' [\Pi(y), \phi(x')] \\
&= \frac{1}{4} \int_{-\infty}^y dy' i\delta(x - y') + \frac{1}{4} \int_{-\infty}^x dx' i\delta(x' - y) \\
&= \frac{i}{4}.
\end{aligned} \tag{3.46}$$

We now want to evaluate some correlation functions of this theory, the same way we did for the fermionic case. The correlation function for the right movers is

$$\begin{aligned}
G_+(x) &\equiv \langle \phi_+(x)\phi_+(0) - \phi_+(0)^2 \rangle \\
&= \int_0^{\infty} \frac{dp}{2\pi\sqrt{2|p|}} \frac{dq}{2\pi\sqrt{2|p|}} e^{-\frac{\alpha}{2}(|q|+|p|)} \langle \phi(p)\phi^\dagger(q) \rangle (e^{ipx} - 1) \\
&= \int_0^{\infty} \frac{dp}{4\pi p} (e^{ipx} - 1) e^{-\alpha p} \\
&= \frac{1}{4\pi} \ln \frac{\alpha}{\alpha - ix}.
\end{aligned} \tag{3.47}$$

For the left movers we have that

$$\begin{aligned}
G_-(x) &\equiv \langle \phi_-(x)\phi_-(0) - \phi_-(0)^2 \rangle \\
&= - \int_0^{-\infty} \frac{dp}{4\pi|p|} (e^{ipx} - 1) e^{-\alpha|p|} \\
&= \int_0^{\infty} \frac{dp}{4\pi p} (e^{-ipx} - 1) e^{-\alpha p} \\
&= \frac{1}{4\pi} \ln \frac{\alpha}{\alpha + ix}.
\end{aligned} \tag{3.48}$$

Also, a correlation function involving the two cases can be defined as

$$\begin{aligned}
G(x) &\equiv \langle \phi(x)\phi(0) - \phi(0)^2 \rangle \\
&= \langle (\phi_+(x) + \phi_-(x)) (\phi_+(0) + \phi_-(0)) - (\phi_+(0) + \phi_-(0))^2 \rangle \\
&= G_+(x) + G_-(x) - \int_0^{-\infty} \int_0^{\infty} \frac{dp}{2\pi|p|} \frac{dq}{2\pi|q|} e^{-\frac{\alpha}{2}(|p|+|q|)} \delta(p-q)(e^{ipx} - 1) \\
&\quad - \int_0^{-\infty} \int_0^{\infty} \frac{dp}{2\pi|p|} \frac{dq}{2\pi|q|} e^{-\frac{\alpha}{2}(|p|+|q|)} \delta(p-q)(e^{iqx} - 1) \\
&= G_+(x) + G_-(x) \\
&= \frac{1}{4\pi} \ln \frac{\alpha^2}{\alpha^2 + x^2}.
\end{aligned} \tag{3.49}$$

In the third passage of equation (3.49), the correlation functions between  $\phi_+\phi_-$  will cancel out each other. That happens because  $q$  and  $p$  have different integration limits, and so the delta function can only be non-zero for  $q = p = 0$ . In this limit, the functions will vanish.

We want to consider a couple more correlation functions, but before that we have to remember two important identities. Considering two operators  $A$  and  $B$  whose commutation relation between them is a c-number, then

$$e^A e^B =: e^{A+B} : e^{\langle AB + \frac{A^2+B^2}{2} \rangle}, \tag{3.50}$$

and

$$\langle : e^A : \rangle = 1, \tag{3.51}$$

where  $: A :$  means the normal ordering of the operator  $A$  and we are considering  $A$  to be a non-empty product of annihilation and creation operators. When we use the normal order, all the destruction operators are placed at the right and all the creation operator are placed on the left. Having those identities, it is now possible to evaluate some quantities that may appear in two space-time dimensional theories, such as

$$\begin{aligned}
G_\beta(x) &\equiv \langle e^{i\beta\phi(x)} e^{-i\beta\phi(0)} \rangle \\
&= \langle : e^{i\beta\phi(x) - i\beta\phi(0)} : \rangle e^{\beta^2 \langle \phi(x)\phi(0) - \phi(0)^2 \rangle} \\
&= e^{\beta^2 G(x)} \\
&= e^{\beta^2 \frac{1}{4\pi} \ln \frac{\alpha^2}{\alpha^2 + x^2}} \\
&= \left( \frac{\alpha^2}{\alpha^2 + x^2} \right)^{\frac{\beta^2}{4\pi}}.
\end{aligned} \tag{3.52}$$

If we set  $\alpha$  to zero on equation (3.52), the correlation function also vanishes. A way to contour this issue will be properly discussed when needed.

There is one last object to be defined. Instead of parameterizing the problem only with the scalar field  $\phi = \phi_+ + \phi_-$ , sometimes it is useful to study the problem in terms of the dual field  $\theta$ , defined as

$$\theta(x) \equiv \phi_-(x) - \phi_+(x), \quad (3.53)$$

or,

$$\theta(x) = \int_{-\infty}^x dx' \Pi(x'). \quad (3.54)$$

The correlation function of the dual field is the same as the field  $\phi$ , that is,

$$\begin{aligned} \langle e^{i\beta\theta(x)} e^{-i\beta\theta(0)} \rangle &= \langle : e^{i\beta\theta(x) - i\beta\theta(0)} : \rangle e^{\beta^2 \langle \theta(x)\theta(0) - \theta(0)^2 \rangle} \\ &= e^{\beta^2 \langle \phi(x)\phi(0) - \phi(0)^2 \rangle} \\ &= e^{\beta^2 G(x)} \\ &= e^{\beta^2 \frac{1}{4\pi} \ln \frac{\alpha^2}{\alpha^2 + x^2}} \\ &= \left( \frac{\alpha^2}{\alpha^2 + x^2} \right)^{\frac{\beta^2}{4\pi}}. \end{aligned} \quad (3.55)$$

In this chapter, we only use the phase field  $\phi$ , but the dual field will be useful for further calculations. The computations in the last two sections basically summarize all we need from the fermionic and bosonic systems. We are now able to create the bosonization dictionary.

### 3.3 BOSONIZATION DICTIONARY

We can now fulfill the goal to express one fermionic theory in terms of a bosonic theory. We will see how to go from one problem to the other and prove that they are equivalent. The most important equation is the bosonization formula, given by

$$\psi_{\pm} = \frac{1}{\sqrt{2\pi\alpha}} e^{\pm i\sqrt{4\pi}\phi_{\pm}}. \quad (3.56)$$

This equation means that we can get the same results from any correlation function of  $\psi$  calculated in the Dirac sea by applying the bosonic operator, placed on the right side of equation (3.56), on the bosonic vacuum with same momentum cut-off. This is a duality, in which a theory has two valid descriptions, where the fundamental particles of one theory (the fermions on the left side) are seen as solitons in the other theory (represented by the operators on the right side) [26]. Instead of doing a formal calculation to obtain equation (3.56), we are going to use it to derive correlation functions and prove that they are equivalent to our previous results.

We start by applying the formula to equation (3.20), which becomes

$$\begin{aligned}
\langle \psi_+(x)\psi_+^\dagger(0) \rangle &= \left\langle \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_+(x)} \frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_+(0)} \right\rangle \\
&= \frac{1}{2\pi\alpha} \left\langle : \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_+(x)} \frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_+(0)} : \right\rangle e^{4\pi\langle \phi_+(x)\phi_+(0) - \phi_+^2(0) \rangle} \\
&= \frac{1}{2\pi\alpha} \frac{1}{-ix}.
\end{aligned} \tag{3.57}$$

Equations (3.47), (3.50) and (3.51) were needed to perform this calculation. The result obtained is exactly equal to the one present on equation (3.20). Another common product that usually shows up in the fermionic theory is

$$\begin{aligned}
\bar{\psi}(x)\psi(x) &= \psi^\dagger(x)\gamma^0\psi(x) \\
&= \psi^\dagger(x)\sigma_2\psi(x) \\
&= -i\psi_+^\dagger(x)\psi_-(x) + i\psi_-^\dagger(x)\psi_+(x) \\
&= \frac{1}{2\pi\alpha} \left( -ie^{-i\sqrt{4\pi}\phi_+(x)} e^{-i\sqrt{4\pi}\phi_-(x)} + ie^{i\sqrt{4\pi}\phi_-(x)} e^{i\sqrt{4\pi}\phi_+(x)} \right) \\
&= \frac{1}{2\pi\alpha} \left( -ie^{-i\sqrt{4\pi}\phi(x)} e^{-i\frac{i}{2}\pi} + ie^{i\sqrt{4\pi}\phi(x)} e^{\frac{i}{2}\pi} \right) \\
&= \frac{1}{2\pi\alpha} \left[ - \left( \cos \sqrt{4\pi}\phi(x) - i \sin \sqrt{4\pi}\phi(x) \right) - \left( \cos \sqrt{4\pi}\phi(x) + i \sin \sqrt{4\pi}\phi(x) \right) \right] \\
&= -\frac{1}{\pi\alpha} \cos \sqrt{4\pi}\phi.
\end{aligned} \tag{3.58}$$

To solve this equation, the BCH theorem was used to write down  $\phi_+$  and  $\phi_-$  in terms of  $\phi$ . As the commutator between  $\phi_+$  and  $\phi_-$  is just a number (given by equation (3.46)), the BCH theorem reduces to

$$e^A e^B = e^{A+B} e^{+\frac{1}{2}[A,B]}, \tag{3.59}$$

or

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}. \tag{3.60}$$

Following the same path, we can compute

$$\begin{aligned}
\bar{\psi}(x)i\gamma^5\psi(x) &= \psi^\dagger(x)\gamma^0 i\gamma^5\psi(x) \\
&= \psi(x)^\dagger \sigma_2 i\sigma_3 \psi(x) \\
&= -\psi_+^\dagger(x)\psi_-(x) - \psi_-^\dagger(x)\psi_+(x) \\
&= \frac{1}{\pi\alpha} \sin \sqrt{4\pi}\phi(x).
\end{aligned} \tag{3.61}$$

There is a possible problem that we have to consider when dealing with a product of operators

in the same point (like in equation (3.61)). A product of two well-behaved operators in the same point may not be well-defined when computing the expectation value of this quantity, for example  $\langle \bar{\psi}(x)\psi(x) \rangle$ . A way to test if the mean value of a product of operators will be non-zero is to check if they have a finite matrix element when the arguments approach each-other. We can see the application in the following equation,

$$\begin{aligned}
\psi_+^\dagger(0)\psi_+(0) &= \lim_{x \rightarrow 0} \frac{1}{2\pi\alpha} e^{-i\sqrt{4\pi}\phi_+(x)} e^{i\sqrt{4\pi}\phi_+(0)} \\
&= \lim_{x \rightarrow 0} \frac{1}{2\pi\alpha} : e^{\sqrt{4\pi}(-i\phi_+(x)+i\phi_+(0))} : e^{4\pi G_+(x)} \\
&= \lim_{x \rightarrow 0} \frac{1}{2\pi\alpha} : e^{\sqrt{4\pi}\left(-i\frac{\partial\phi_+(x)}{\partial x}x\right)} : e^{4\pi G_+(x)} \\
&= \lim_{x \rightarrow 0} \frac{i}{2\pi(x+i\alpha)} : 1 - i\sqrt{4\pi}\frac{\partial\phi_+(x)}{\partial x}x + \dots : \\
&= \lim_{x \rightarrow 0} \frac{i}{2\pi x} + \frac{1}{\sqrt{\pi}} \frac{\partial\phi_+(x)}{\partial x} + \dots
\end{aligned} \tag{3.62}$$

Some identities and simplifications were used to develop equation (3.62). First we used the identity on equation (3.50). This was necessary because the Taylor expansion is only well defined for normal ordered operators (or else, how would we know if, for example,  $(A+B)^2$  is  $(A+B)(A+B)$  or  $(A+B)(B+A)$ ?). Done that, we applied the Taylor expansion considering  $x$  to be small. Next we used the exponential expansion and, again, considered  $x$  to be small so we can drop terms of higher orders. Finally we also considered  $\alpha$  to be close to zero as we defined in the earlier sections. The last passage in equation (3.62) is a little complicated to understand because we considered that  $\alpha$  goes to zero, even though  $x$  also is consider to be close to zero. This holds because we can treat  $x$ , in the continuum space, to be always larger than  $\alpha$  that is set to zero whenever as possible. If we consider the normal ordering of the operator, we are able to get rid of the first divergent term. So we have

$$: \psi_+^\dagger(x)\psi_+(x) : = \frac{1}{\sqrt{\pi}} \frac{\partial\phi_+}{\partial x}. \tag{3.63}$$

Combining equations (3.62) and (3.63), we can substitute them in equations (3.28) and (3.29) to get

$$j_0 = \frac{1}{\sqrt{\pi}} \frac{\partial\phi_+}{\partial x}, \tag{3.64}$$

and

$$\begin{aligned}
j_1 &= \frac{1}{\sqrt{\pi}} \frac{\partial(\phi_+ - \phi_-)}{\partial x} \\
&= -\frac{1}{\sqrt{\pi}} \frac{\partial\theta}{\partial x} \\
&= -\frac{\Pi}{\sqrt{\pi}}.
\end{aligned} \tag{3.65}$$

These two equations can be combined in one by using the covariant notation

$$j_\mu = \frac{\epsilon_{\mu\nu}}{\sqrt{\pi}} \partial_\nu \phi. \tag{3.66}$$

All the results obtained with the bosonization formula matches the previous calculations done in sections (3.1) and (3.2), proving that the bosonization formula is valid. Also, substituting the fermionic operators for the respective bosonic operators, it can be proven that

$$\begin{aligned}
H_F &= \int \frac{dp}{2\pi} \psi_+^\dagger(p)(p)\psi_+(p) + \psi_-^\dagger(p)(-p)\psi_-(p) \\
&= \int \frac{dp}{2\pi} |p| \phi^\dagger(p)\phi(p) = H_B.
\end{aligned} \tag{3.67}$$

Our previous results and some new ones from references [4, 3] are summarize below.

$$\bar{\psi}\psi \rightarrow \frac{1}{2}(\partial_\mu\phi)^2 = \frac{1}{2}[(\partial_t\phi)^2 - (\partial_x\phi)^2], \tag{3.68}$$

$$\bar{\psi}\gamma^\mu\psi \rightarrow j_\mu = \frac{\epsilon_{\mu\nu}}{\sqrt{\pi}}\partial^\nu\phi, \tag{3.69}$$

$$\bar{\psi}\psi \rightarrow -\wedge \cos\sqrt{4\pi}\phi, \tag{3.70}$$

$$\bar{\psi}i\gamma^5\psi \rightarrow \wedge \sin\sqrt{4\pi}\phi, \tag{3.71}$$

$$(\bar{\psi}\psi)^2 = \left[-\wedge \cos\sqrt{4\pi}\phi\right]^2 = -\frac{1}{2\pi}(\partial^\mu\phi\partial_\mu\phi). \tag{3.72}$$

In the above,  $\wedge$  is a momentum cut-off defined as

$$\frac{1}{\pi\alpha} \equiv \wedge, \tag{3.73}$$

where  $\alpha$  was the spacial momenta cut-off. It is now useful to see some systems in which we can use the bosonization technique and the dictionary.

## 4 APPLICATIONS OF BOSONIZATION

We may now see how the bosonization formulas can be applied to solve a few models to get more familiarity with the method. We are not going to develop all the calculation of these models, only relevant points to understand how bosonization can be implemented in a theory. The renormalization group reviewed in chapter 2 will also be useful. We shall consider first the massless Schwinger and Thirring models, because the Dirac fermions are already present in the definition of their Lagrangians [3].

### 4.1 MASSLESS SCHWINGER MODEL

The model was introduced by Schwinger [27], and it describes the two dimensional electrodynamics, given by the Lagrangian

$$\mathcal{L} = \bar{\psi}\not{\partial}\psi - j^\mu A_\mu + \frac{1}{2}(\epsilon^{\mu\nu}\partial_\mu A_\nu)^2, \quad (4.1)$$

where  $A_\nu$  is the vector potential,  $\epsilon^{\mu\nu}$  is the Levi-Civita symbol and  $j^\mu$  is the current density. The equation of motion for the field  $A_\mu$  gives

$$j^\mu = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu). \quad (4.2)$$

We now want to use the bosonization equations. We can use equation (3.69) on the current term to get

$$j^\mu A_\mu = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi A_\mu = -\phi\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu A_\mu, \quad (4.3)$$

such that we performed an integration by parts and discarded a total derivative term. Replacing equations (3.68) and (4.3) in (4.1), plus completing the square, we have

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}(\epsilon_{\mu\nu}\partial^\mu A^\nu + \frac{\phi}{\sqrt{\pi}})^2 - \frac{\phi^2}{2\pi}. \quad (4.4)$$

Using equations (3.69) and (4.2), we are left with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\phi^2}{2\pi}. \quad (4.5)$$

The Lagrangian described in equation (4.4) has the momentum propagator, in the Minkowski

space, given by

$$G_F(k) = \frac{-i}{k^2 - \frac{1}{\pi}}. \quad (4.6)$$

This means that there is a pole in

$$k^2 = m^2 = \frac{1}{\pi}. \quad (4.7)$$

There are some points worth noticing. First we began with a theory that seemed massless. After applying the bosonization equations, we end up with a bosonic system with an explicit massive term, indicating that the original fermionic system is massive when we computed the propagator of the theory. Due to the electromagnetic interaction, the field  $A$  acquired a mass by what it is called the Schwinger mechanism [28]. Schwinger argued that gauge invariance present in equation (4.1) was not able to guarantee a massless electromagnetic field [27]. The main point is that after the bosonization procedure, it becomes quite clear that the system is massive, because of the term  $\frac{\phi^2}{2\pi}$ , which demonstrate how the work can become much simpler in the bosonic language. Let us move on to the next problem.

## 4.2 MASSLESS THIRRING MODEL

Another model that already contains Dirac fermions is the massless Thirring model, first introduced by Thirring in 1964 [6]. The model is represented by

$$\mathcal{L} = \bar{\psi}\not{\partial}\psi + \frac{g}{2}j^\mu j_\mu, \quad (4.8)$$

where  $g$  is a dimensionless coupling constant and  $\psi$  has dimension  $\frac{1}{2}$  in mass units. After applying the bosonization formulas (3.68) and (3.69), the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 \left(1 + \frac{g}{2}\right). \quad (4.9)$$

Now we are able to estimate correlation functions of this model, in a simpler way, which is our main goal. Using dimensional analysis, a quartic correlation function of the non-interacting fermionic theory is

$$\langle \bar{\psi}(r)\psi(r)\bar{\psi}(0)\psi(0) \rangle \simeq \frac{1}{r^2}, \quad (4.10)$$

because each field  $\psi$  has spatial dimension of  $-\frac{1}{2}$ , so the right side of equation (4.10) must be proportional to  $\frac{1}{r^2}$ . We want to obtain this correlation function by using the bosonization

technique. Using equation (3.70) into equation (4.10), we obtain

$$\begin{aligned}
\langle \bar{\psi}(r)\psi(r)\bar{\psi}(0)\psi(0) \rangle &= \frac{1}{\pi^2\alpha^2} \left\langle \cos \sqrt{4\pi}\phi(r) \cos \sqrt{4\pi}\phi(0) \right\rangle \\
&= \frac{1}{2\pi^2\alpha^2} \left\langle \left( e^{i \cos \sqrt{4\pi}\phi(r)} + e^{-i \cos \sqrt{4\pi}\phi(r)} \right) \left( e^{i \cos \sqrt{4\pi}\phi(0)} + e^{-i \cos \sqrt{4\pi}\phi(0)} \right) \right\rangle \\
&= \frac{1}{2\pi^2\alpha^2} \left( \frac{\alpha^2}{r^2} \right)^{\frac{4\pi}{4\pi}} \simeq \frac{1}{r^2}.
\end{aligned} \tag{4.11}$$

In order to solve this equation we used equation (3.70) and later applied equation (3.52) in the correlation function of the scalar field. We need to be careful when doing this naive substitutions for the interacting system in (4.9). These formulas were developed for a theory with a kinetic term of the form  $\frac{1}{2}(\nabla\phi)^2$ . Here we have  $(\frac{1}{2} + \frac{g}{2\pi})(\nabla\phi)^2$ . Instead of recalculating the correlation function for a interaction, we can redefine the field as

$$\phi' = \sqrt{1 + \frac{g}{\pi}}\phi, \tag{4.12}$$

so the answers we want come as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi')^2, \tag{4.13}$$

$$\bar{\psi}\psi = -\frac{1}{\pi\alpha} \cos \sqrt{\frac{4\pi}{1 + \frac{g}{\pi}}}\phi', \tag{4.14}$$

and

$$\langle \bar{\psi}(r)\psi(r)\bar{\psi}(0)\psi(0) \rangle = \frac{1}{2\pi^2\alpha^2} \left( \frac{\alpha^2}{r^2} \right)^{\frac{4\pi}{4\pi(1 + \frac{g}{\pi})}} \simeq \frac{1}{r^{\frac{2}{1 + \frac{g}{\pi}}}}. \tag{4.15}$$

Analyzing the equation (4.15), it is possible to see that the correlation function decays with powers of  $g$ . This shows us how the correlation function changes in the presence of the interaction. Also, equation (4.15) displays the need of the cut-off to be introduced (in this case  $\alpha$ ), as it serves as an additional dimensional parameter [3]. The main point of these calculations is that we went from an interacting theory Lagrangian, in equation (4.8), to a free theory Lagrangian, in (4.13), after applying the bosonization equations. This represents a great simplification of our system, once again showing how the bosonization technique can be useful.

The Thirring model was a good example to implement the bosonization formulas, even though the system did not seem to have the exact needed form. For the next sections we go even deeper. We want to obtain the Sine-Gordon Hamiltonian and the Tomonaga-Luttinger liquid via bosonization and then apply the renormalization group in the system. For this purpose we start with a Non-Relativistic Lattice fermion model.

### 4.3 NON-RELATIVISTIC LATTICE FERMIONS IN ONE DIMENSION

A non-relativistic fermion lattice is composed by spinless fermions in a one-dimensional lattice with an interaction [9]. This structure is represented by the Hamiltonian

$$\begin{aligned} H &= H_0 + H_I \\ &= -\frac{1}{2} \sum_j \psi^\dagger(j+1)\psi(j) + \psi^\dagger(j)\psi(j+1) + \Delta \sum_j \left( \psi^\dagger(j)\psi(j) - \frac{1}{2} \right) \left( \psi^\dagger(j+1)\psi(j+1) - \frac{1}{2} \right), \end{aligned} \quad (4.16)$$

where  $j$  stands for the site of the fermion;  $H_0$  stands for the free part of the Hamiltonian, composed by a hopping term represented in the first sum; and  $H_I$  stands for the interaction part of the Hamiltonian, given by an interaction term between nearest-neighbor fermions. We want to consider positive values of  $\Delta$ , which represents a repulsive interaction as we will see shortly. The fermions fields respect the anti-commutation rule

$$\{\psi^\dagger(j), \psi(j')\} = \delta_{jj'}, \quad (4.17)$$

with other anti-commutation rules vanishing [9].

The  $-\frac{1}{2}$  terms present in the Hamiltonian are needed to maintain the particle-hole symmetry of the system, represented by the exchange  $\psi(j) \rightarrow \psi^\dagger(j)$  in the Hamiltonian. If we perform this transformation in  $H$ , the number operator  $n = \psi^\dagger(j)\psi(j)$  becomes  $n-1$ , but because of the  $-\frac{1}{2}$  terms, the Hamiltonian remains invariant. The system would already have this symmetry if it was not for the interaction term, thus the need of the fractional factors. This also implies that

$$\langle n \rangle = \langle 1 - n \rangle, \quad (4.18)$$

and so

$$\langle n \rangle = \frac{1}{2}. \quad (4.19)$$

This result means that we are considering the half-filling of the system, where half of the possible states are occupied.

Having the Hamiltonian of the system, we want to analyze two asymptotic limits of the interaction term,  $\Delta \rightarrow 0$  and  $\Delta \rightarrow \infty$ . Let us evaluate the first case. For weak coupling, we begin introducing the momentum states via

$$\psi(j) = \int_{-\pi}^{\pi} \frac{dK}{2\pi} \psi(K) e^{iKj}. \quad (4.20)$$

Since we are in a lattice, the particles are periodically arranged and, therefore, we limited our

integration from  $-\pi$  to  $\pi$  to restrict the calculation to the first Brillouin zone, which consists of the unit cell that is repeated to build up the lattice [29]. All the possible energy values for the electron are contained in this zone, so there is no need to extrapolate this limit. Plugging equation (4.20) into  $H$ , we get

$$\begin{aligned}
 H = H_0 &= -\frac{1}{8\pi^2} \sum_j \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dk dk' \left( \psi^\dagger(k) \psi(k') e^{-ij(k-k')-ik} + \psi^\dagger(k) \psi(k') e^{-ij(k-k')+ik'} \right) \\
 &= -\frac{1}{8\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dk dk' \left( 2\pi \delta(k-k') \psi^\dagger(k) \psi(k') e^{-ik} + 2\pi \delta(k-k') \psi^\dagger(k) \psi(k') e^{ik'} \right) \\
 &= -\frac{1}{4\pi} \int_{-\pi}^{\pi} dk \left( \psi^\dagger(k) \psi(k) e^{-ik} + \psi^\dagger(k) \psi(k) e^{ik} \right) \\
 &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \psi^\dagger(k) \psi(k) \cos(k), \tag{4.21}
 \end{aligned}$$

$$E(K) = -\cos K, \tag{4.22}$$

where  $E(K)$  is the energy density of the fermions, and it was used the fact that

$$\sum_j e^{ij(K-K')} = 2\pi \delta(K-K'). \tag{4.23}$$

To interpret the result obtained, it may be helpful to keep figure (4.1) in mind. The Fermi sea is filled with all the negative energy states, those with  $|K| \leq K_F = \frac{\pi}{2}$ , which correspond to half-filling of the system. We could also add a constant to our Hamiltonian, so the lowest energy state for  $k = 0$  had zero energy and all the states had positive energy, but what is required is that our system is bounded from below, so there is no problem with our definition. The Fermi surface is then consisted just by the points  $|K| = \pm \frac{\pi}{2}$ . At the ground state, if we give an infinitesimal excitation for a particle, the energy of the system changes linearly, that is

$$K_F \rightarrow K_F + \delta K, \tag{4.24}$$

$$E(K_F) = 0 \rightarrow E(K_F + \delta K) = \sin(\delta K) = \delta K. \tag{4.25}$$

This means that there is no energy gap and the particle can move freely in the ground state.

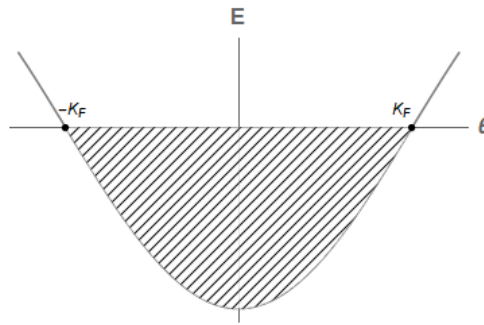


Figura 4.1: This graphic represents the half-filling of the system, where the covered region is the Fermi sea filled with states with negative energies. The Fermi surface is reduced to only two points, represented by  $K_F$  and  $-K_F$ , where  $K_F = \frac{\pi}{2}$ .

The second case,  $\Delta \rightarrow \infty$ , is simpler. Looking at the quartic term in the Hamiltonian, we can see that for positive values of  $\Delta$ , the system tends to minimize the value of the operators  $\psi^\dagger \psi$  either for the site  $j$  or the site  $j + 1$ , to minimize the energy. This indicates that our interaction is repulsive. If the interaction is large enough, either the field operators  $\psi^\dagger(j)\psi(j)$  or  $\psi^\dagger(j+1)\psi(j+1)$  in equation (4.16) must vanish. As these operators represent the occupation numbers, the particles would occupy one or other sublattice, and so every particle would have no neighbor [3]. All the even sites or all the odd sites would be occupied, two structures that are called *Charge Density Wave* or CDW states, represented by figure (4.2). A charge density wave represents a modulation of conduction electrons density, creating an energy gap at the Fermi surface [30]. This means that any excitation of the ground state, which requires us to move the particles, now costs an energy of the order of the gap ( $\Delta$ ).

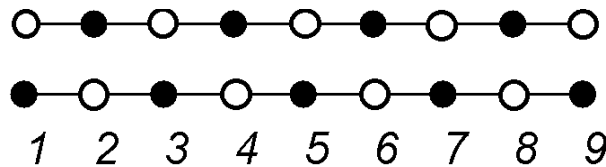


Figura 4.2: The two possible CDW states. A filled circle means the the site is occupied by a fermion while an empty circle represents an empty site. The first line represents a state where all the even sites of the lattice are occupied and the second line represents a state with all the odd sites occupied.

The question to be answered is: if we have a weak interaction, the system would develop the CDW gap or it would remain on its fixed line?. We shall use bosonization and the RG group to get some of the wanted results, and while analyzing this types of interactions, we will find out the Sine-Gordon Hamiltonian and the Tomonaga-Luttinger Liquid.

#### 4.4 DERIVING THE SINE-GORDON HAMILTONIAN

We want Dirac fermions to use bosonization. They will appear in the theory when we consider the continuum limit and a low-energy limit [31]. We introduce the lattice

spacing parameter  $a$  with the purpose of making the  $j$  index continuum. Also, to achieve the low-energy limit, we consider only the states in a neighborhood of the ground state, that is, states with  $K = K_F \pm \lambda$  and  $K = -K_F \pm \lambda$  with  $\lambda$  being a small variation. Keeping this restrictions in mind, the Fourier transform becomes

$$\begin{aligned}
\psi(j) &= \int_{-\pi}^{\pi} \frac{dK}{2\pi} \psi(K) e^{iKj} \\
&\simeq \int_{-\lambda}^{\lambda} \frac{dk}{2\pi} \psi(K_F + k) e^{iK_F j} e^{ikj} + \int_{-\lambda}^{\lambda} \frac{dk}{2\pi} \psi(-K_F + k) e^{-iK_F j} e^{-ikj} \\
&\equiv a^{\frac{1}{2}} \left[ e^{iK_F j} \psi_+(x = aj) + e^{-iK_F j} \psi_-(x = aj) \right] \\
&= a^{\frac{1}{2}} \left[ e^{i\frac{\pi}{2}j} \psi_+(x = aj) + e^{-i\frac{\pi}{2}j} \psi_-(x = aj) \right]. \tag{4.26}
\end{aligned}$$

Now, in the momentum space, our fields have the same spectrum of figure (3.1). We once again have to introduce the Dirac sea so the movers are well defined. While it may seem that we are introducing "unphysical" states into our theory, in the vicinity of the Fermi momenta these states play no role in our system, as we would need a large amount of energy to excite them from the negative-energy states to the Fermi surface.

Now we replace the low-energy Dirac field, derived in equation (4.26), in the Hamiltonian. Thus,

$$\begin{aligned}
H_0 &= -\frac{1}{2} \sum_j \psi^\dagger(j+1)\psi(j) + \psi^\dagger(j)\psi(j+1) \\
&= -\frac{1}{2} a \sum_j \left[ -ie^{-i\frac{\pi}{2}j} \psi_+^\dagger(x = ja + a) + ie^{i\frac{\pi}{2}j} \psi_-^\dagger(x = ja + a) \right] \\
&\quad \times \left[ e^{i\frac{\pi}{2}j} \psi_+(x = ja) + e^{-i\frac{\pi}{2}j} \psi_-(x = ja) \right] + h.c.. \tag{4.27}
\end{aligned}$$

After some simple algebra, dropping oscillating terms at  $\pm K_F j$ , and an integral by parts, we can define

$$H_{0c} = \frac{H_0}{a} = \int dx \left[ \psi_+^\dagger(x) (-i\partial_x) \psi_+(x) + \psi_-^\dagger(x) (i\partial_x) \psi_-(x) \right], \tag{4.28}$$

where we considered  $a \sum_j \rightarrow \int dx$ , and  $H_{0c}$  is the non-interacting continuum Hamiltonian.

The same procedure can be done for the interacting Hamiltonian, that is,

$$\begin{aligned}
H_I &= \Delta \sum_j \left( \psi^\dagger(j)\psi(j) - \frac{1}{2} \right) \left( \psi^\dagger(j+1)\psi(j+1) - \frac{1}{2} \right) \\
&= \Delta \sum_j : \psi^\dagger(j)\psi(j) :: \psi^\dagger(j+1)\psi(j+1) : . \tag{4.29}
\end{aligned}$$

In the above it was taken into account that

$$: \psi^\dagger(j)\psi(j) := \psi^\dagger(j)\psi(j) - \langle \psi^\dagger(j)\psi(j) \rangle = \psi^\dagger(j)\psi(j) - \frac{1}{2}. \quad (4.30)$$

This property is valid because of the half-filling of the system. Applying equations (4.26), (3.63), (3.70) and (3.72) to equation (4.29), up to oscillatory terms, we have

$$H_{Ic} = \frac{H_I}{a} = \Delta \int dx \left[ \frac{2(\partial_x \phi)^2}{\pi} + \frac{1}{2\pi^2 \alpha^2} \cos(\sqrt{16\pi}\phi) \right]. \quad (4.31)$$

We are able to write the full Hamiltonian given by equations (4.31) plus (4.28) in the bosonized form as

$$\begin{aligned} H_c K &= H_{0c} + H_{Ic} = \int dx \left[ \frac{1}{2} \left( \Pi^2 + \left(1 + \frac{4\Delta}{\pi}\right) (\partial_x \phi)^2 \right) + \frac{\Delta}{2\pi^2 \alpha^2} \cos(\sqrt{16\pi}\phi) \right] \\ &= \int dx \left[ \frac{1}{2} \left( K\Pi^2 + \frac{1}{K} (\partial_x \phi)^2 \right) + \frac{y}{2\pi^2 \alpha^2} \cos(\sqrt{16\pi}\phi) \right], \end{aligned} \quad (4.32)$$

such that  $K$  is the Luttinger parameter and we introduced a change of variable, represented by  $y$ . These new variables are

$$K = \left[ 1 + \frac{4\Delta}{\pi} \right]^{\frac{1}{2}}, \quad (4.33)$$

and

$$y = K\Delta = \frac{\Delta}{\sqrt{1 + \frac{4\Delta}{\pi}}}. \quad (4.34)$$

We will consider  $y$  and  $K$  two independent parameters of the theory, rather than functions of  $\Delta$  [3]. We can now rescale  $H_c$  by defining a new field and momentum

$$\phi' = \frac{1}{\sqrt{K}}\phi, \quad (4.35)$$

$$\Pi' = \sqrt{K}\Pi. \quad (4.36)$$

Then after dropping primes, the Hamiltonian becomes

$$H_c = \int dx \left[ \frac{1}{2} (\Pi^2 + (\partial_x \phi)^2) + \frac{y}{2\pi^2 \alpha^2} \cos(\sqrt{16K\pi}\phi) \right]. \quad (4.37)$$

Performing a variable change in equation (4.37), we get the Sine-Gordon Ha-

miltonian, given by

$$H_{SG} = \int dx \left[ \frac{1}{2} (\Pi^2 + (\partial_x \phi)^2) + \frac{y}{2\pi^2 \alpha^2} \cos(\beta \phi) \right], \quad (4.38)$$

with

$$\beta^2 \equiv 16\pi K. \quad (4.39)$$

This system is also called the Luttinger model. We also use a related parameter (called  $x$ , although it does not have anything to do with the integration variable) as

$$x \equiv 2 - 4K = 2 \left( 1 - \frac{\beta^2}{8\pi} \right). \quad (4.40)$$

Having the bosonic Hamiltonian, we shall remember what our goals are. We wanted to define the Sine-Gordon model from a fermion system, using the bosonization formulas. We reached this objective by starting with a one space dimension fermion system, and got a Hamiltonian that describes a boson field with an interaction, given by equation (4.38). Our parameters now are  $y$  and  $K$ , where the last one is yet related to  $\beta$  and  $x$ . Now we plan to see how the interaction term affects the bosonic system. We will apply the renormalization group to the action of the system in 1+1 dimensions to figure out how this interaction contribute to the system and finally answer the proposed question at the end of section 4.3.

#### 4.4.1 Renormalization of the Sine-Gordon Model

Considering that  $\Pi^2 - \partial_x \phi^2 \equiv (\nabla \phi)^2$ , the action of the system in 1+1 dimensions is

$$S = \int d^2x \left( \frac{1}{2} (\nabla \phi)^2 - \frac{y \Lambda^2}{2} \cos(\beta \phi) \right), \quad (4.41)$$

where  $d^2x$  is over the space and time parameters. In this equation, we made the replacement  $\frac{1}{\pi \alpha} = \Lambda$ . This will play a key role in defining integration limits.

We must take the low-energy limit to see how the interaction term acts on the system, so it would be interesting if we can integrate out the fast modes over the path integral, that means, a thin shell slightly below the momentum cut-off  $\Lambda$ . Let us write  $\phi$  as a sum of slow and fast modes

$$\phi = \phi_s + \phi_f = \phi(0 \leq k \leq \Lambda(1 - d\mu)) + \phi(\Lambda(1 - d\mu) < k \leq \Lambda), \quad (4.42)$$

being  $d\mu$  a small dimensionless parameter. The free action represented by the first term in

equation (4.41) becomes

$$S_0 = \int d^2x \left[ \frac{1}{2}(\nabla\phi_s)^2 + \frac{1}{2}(\nabla\phi_f)^2 \right]. \quad (4.43)$$

We do not consider the crossed terms because, when we expand the bosonic fields in fourier modes, the slow modes and the fast modes will have different momentum integration limits. The  $x$  integration will generate a delta function between these momenta, which makes products of  $\phi_s\phi_f$  and  $\phi_f\phi_s$  vanish.

The RG that leaves  $S_0$  invariant involving integrating out  $\phi_f$  and rescaling of the spacetime coordinates. The transformations are

$$d^2x = \frac{1}{s^2}d^2x', \quad (4.44)$$

$$\frac{d}{dx} = s\frac{1}{dx'}, \quad (4.45)$$

and

$$\phi(x) = \phi'(x), \quad (4.46)$$

where we take  $s$  to be  $1 - d\mu$ .

Now  $\phi_f$  can be integrated in the path integral  $Z$ . It is then possible to analyze how the interaction term evolves with the renormalization. Replacing the transformations (4.44), (4.45) and (4.46) into  $Z$  we have

$$\begin{aligned} Z &= \int d\phi_s d\phi_f e^{iS(\phi)} = \int d\phi_s \int d\phi_f \exp \left[ i \int d^2x \frac{1}{2}(\nabla\phi_s)^2 + \frac{1}{2}(\nabla\phi_f)^2 - \frac{y\Lambda^2}{2} \cos(\beta(\phi_s + \phi_f)) \right] \\ &= \int d\phi_s \exp \left[ i \int d^2x \frac{1}{2}(\nabla\phi_s)^2 \right] \left\langle \exp \left[ -i \frac{y\Lambda^2}{2} \int d^2x \cos(\beta(\phi_s + \phi_f)) \right] \right\rangle_f \\ &= \int d\phi_s \exp \left[ i \int \left[ d^2x \frac{1}{2}(\nabla\phi_s)^2 - \frac{y\Lambda^2}{2} \cos(\beta\phi_s) \langle \cos(\beta\phi_f) \rangle_f \right] \right], \end{aligned} \quad (4.47)$$

where  $\langle \dots \rangle_f$  means the expected value over the fast modes. In the last step it was used the approximation  $\langle e^A \rangle = e^{\langle A \rangle}$ . Also when considering the expansion of  $\cos(\beta(\phi_s + \phi_f))$ , the expected value of  $\sin(\phi_s) \langle \sin(\phi_f) \rangle_f$  is 0. To evaluate the remaining cosine term in equation (4.47) we can use equation (3.50) with  $A = i\beta\phi_f$  and  $B = 0$ , so

$$\langle e^{i\beta\phi_f} \rangle_f = e^{-\frac{1}{2}\beta^2 \langle \phi_f^2 \rangle_f}, \quad (4.48)$$

and

$$\begin{aligned}\langle \cos(\beta\phi_f) \rangle_f &= e^{-\frac{1}{2}\beta^2 \langle \phi_f^2 \rangle_f} = \exp \left[ \frac{-\beta^2}{2} \int_{\Lambda(1-d\mu)}^{\Lambda} \int_0^{2\pi} \frac{kdkd\theta}{4\pi^2} \frac{1}{k^2} \right] \\ &= 1 - \frac{\beta^2}{4\pi} d\mu.\end{aligned}\quad (4.49)$$

Integrated the fast modes, we can rescale the coordinates as described by equation (4.44), so

$$d^2x = \frac{1}{s^2} d^2x' = (1 + 2d\mu) d^2x'. \quad (4.50)$$

This rescale maintains the first term in equation (4.47) invariant. Applying this equation to the partition function, up to dropping primes, we have

$$Z = \int d\phi \exp \left[ i \int \left[ d^2x \frac{1}{2} (\nabla\phi)^2 - \frac{y\Lambda^2}{2} \left( 1 + \left( 2 - \frac{\beta^2}{4\pi} \right) d\mu \right) \cos(\beta\phi) \right] \right]. \quad (4.51)$$

Comparing (4.51) to equation (4.42), we can see that the interaction term evolves as

$$\frac{y\Lambda^2}{2} \int d^2x \cos(\beta\phi) \rightarrow \frac{y\Lambda^2}{2} \left( 1 + \left( 2 - \frac{\beta^2}{4\pi} \right) d\mu \right) \int d^2x \cos(\beta\phi). \quad (4.52)$$

Deriving both sides implies that

$$\frac{dy}{d\mu} = \left[ 2 - \frac{\beta^2}{4\pi} \right] y \quad (4.53)$$

$$= (2 - 4k)y, \quad (4.54)$$

$$\beta^2 = 16\pi K. \quad (4.55)$$

Analyzing equations (4.53)-(4.55) it can be identified that the parameter  $y$  grows or decreases based on the value of  $K$ . This means that the interaction is irrelevant when  $K > \frac{1}{2}$  and relevant when  $K < \frac{1}{2}$ . There is a especial case when  $K > \frac{1}{2}$  and  $y = 0$ , called the Tomonaga-Luttinger Liquid [31]. For these values the perturbation term becomes irrelevant and the system remains in its fixed line, where the renormalization does not create the CDW states. In a Cartesian  $(x, y)$  plane it can be represented by the region

$$y = 0 \quad (4.56)$$

$$x \equiv 2 - 4K < 0. \quad (4.57)$$

Not only the Tomonaga-Luttinger line  $\{y = 0, x < 0\}$  describes the models where  $y = 0$ , but it also describes the models that flow to  $y = 0$  under RG. After these computations we are ready to construct a flow in the  $(x, y)$  plane to analyze how a system would respond to the RG group,

based on the parameter  $K$ .

#### 4.4.2 Renormalization Group Flow

We want to evaluate when the gapped phase will appear after crossing the TL fixed line and analyze when it happens. For  $y$  we have the equation

$$\frac{dy}{d\mu} = (2 - 4K)y = xy. \quad (4.58)$$

While  $x$  is negative, the interaction is irrelevant and the flow tend to the TL line. When  $x$  becomes positive ( $K < \frac{1}{2}$ ), the flow changes its sign and start to grow. This means that this area is tending away from the TL line. What we need is an equation that dictates the evolution of  $x$ .

The only way to renormalize  $K$ , and therefore  $x$ , is by field renormalization, which emerges when we compute correlation functions such as

$$\left\langle \frac{1}{2\pi\alpha} e^{i\sqrt{4\pi}\phi_+(x_1)} e^{-i\sqrt{4\pi}\phi_+(x_2)} \right\rangle_{S_{SG}} = \left\langle \frac{1}{2\pi\alpha} e^{i\sqrt{4\pi}\phi_+(x_1)} e^{-i\sqrt{4\pi}\phi_+(x_2)} e^{-i \int d^2x (\frac{y\Lambda^2}{2} \cos(\beta\phi))} \right\rangle_{S_{0c}}. \quad (4.59)$$

We used equation (2.42) in this step. The flow represented by equation (4.58) is quadratic in small quantities and so should be the  $x$  flow [3]. Considering that  $y$  is a small parameter, and the only contributions of the cosine term after we expand it will be in even order, the first non-zero contribution will be a quadratic term in  $y$ . This means that  $\beta^2$  gets a  $y^2$  term that would be absorbed by renormalization. As  $x$  and  $\beta$  are related by equation (4.40),  $x$  also gains this term. A complete discussion about this flow is given in reference [9], section 4.6. With these arguments in mind plus dropping constants, the  $x$  flow is

$$\frac{dx}{d\mu} = y^2. \quad (4.60)$$

Equations (4.58) and (4.60) represent the flow for the Sine-Gordon model, called the Kosterlitz-Thouless Flow, represented by figure (4.3).

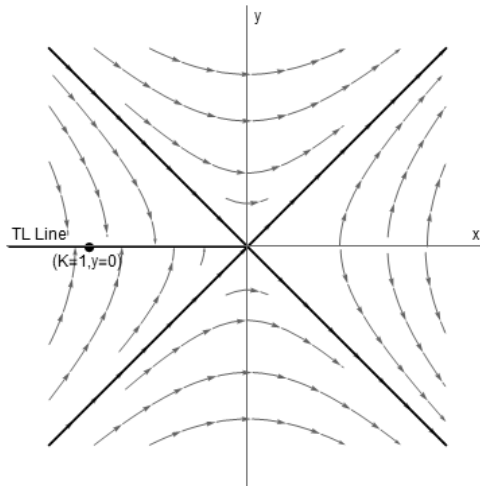


Figura 4.3: The Kosterlitz-Thouless Flow. In the graphic, we can see the Tomonaga-Luttinger Line, the fixed points line representing our gapless system. The point  $(K = 1, y = 0)$  represents the free massless fermion theory.

The KT flow shows all we need to know about the gapped states. If we begin in the second quadrant between the two separatrices, the points tend to the TL line. In the rest of the graphic the flow tends away from this fixed line, converging into the two separatrices  $x = -y$  and  $x = y$ , resulting in gapped phases. The phase-transition happens when the system crosses over this lines, where the correlation function diverges. Also these separatrices represent different physics, as the interaction can be either positive giving the CDW phase, or negative giving a different gapped state. The main point here is that we can define a theory, find the correspondent point in the graphic and know how it will evolve in the renormalization flow. We can easily say if the interaction of the system is relevant or irrelevant, which was our main goal in this section. The next chapter is devoted to a new theme that presents important physical characteristics, the quantum Hall effect.

## 5 QUANTUM HALL EFFECT

We now want to study the Quantum Hall Effect (QHE). The bosonization technique can be used to study this effect, but first we want to use different approaches in this chapter to gain some knowledge about the problem. We are going use the bosonization technique on the QHE in the next chapter.

The classical Hall effect was first discovered by Edwin Hall in 1879 [32]. The system that exhibits the effect consists in charged particles moving in the presence of a strong magnetic field. This simple model presented important discoveries. V. Klitzing received the Nobel prize for the integer quantum Hall effect in 1985. He discovered the effect in 1980 together with Dorda and Pepper [33]. Our goals in this chapter are first to get some acknowledge about the classical effect and the quantum effect, using a microscopic approach [8, 13, 9, 12]. Then we study the problem with a different approach, the Effective Theory, which consists of proposing a quantum field theory to describe the quantum Hall effect based on general properties that our system may has at a low-energy limit [8, 34, 35].

### 5.1 THE CLASSICAL HALL EFFECT

Before starting to solve the Landau Problem, which has a quantum nature, it is important to understand why the Hall system generates a quantum effect. The Hall system consists in a 2-dimensional plane filled with charged particles in the presence of a strong magnetic field  $\vec{B}$  perpendicular to the plane. If we apply an electric field in the  $X$  direction to the system, it is expected to see a  $I_x$  current in the  $X$  direction, but we can also measure the Hall voltage  $V_h$  appearing in the  $y$  direction. This is the Hall effect<sup>1</sup>. The system is depicted by Figure 5.1.

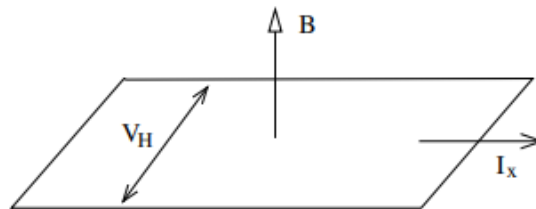


Figura 5.1: Schematic representation of the Classical Hall Effect, where  $B$  is the magnetic field,  $I_x$  is the electric current, and  $V_H$  is the Hall voltage.

Source:Reference [8].

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<sup>1</sup>It is important to notice that the Hall effect appears in the absence of the electrical field, being the dynamics of the electrons still affected by the strong magnetic field. We need the external electrical field to be present so we can analyze how the system reacts under the presence of this field, making possible to measure the Hall voltage.

Following the Drude Model [36], which describes a classical Hamiltonian for the Hall effect, we would expect two resistivities of the form

$$\rho_{xx} = \frac{m_e}{n\tau}, \quad (5.1)$$

$$\rho_{xy} = \frac{B}{n}. \quad (5.2)$$

The first is the resistivity in the  $X$  direction, due to the electric field, and the second is the Hall resistivity in the  $Y$  direction. The parameter  $n$  is the electron density and  $\tau$  is called scattering time, which represents the time between two collisions of electrons.  $m_e$  is the electron mass and  $B$  is the magnetic field. So far we have used a classical model and obtained two continuous resistivities, displayed in the left side of figure 5.2. At low temperatures and strong magnetic fields that is not what really happens. We experimentally see a quantized Hall resistivity, shown in the right side of figure 5.2, where  $\rho \sim \frac{1}{\nu}$  and  $\nu \in \mathbb{Z}$ , being  $\nu$  called filling fraction.

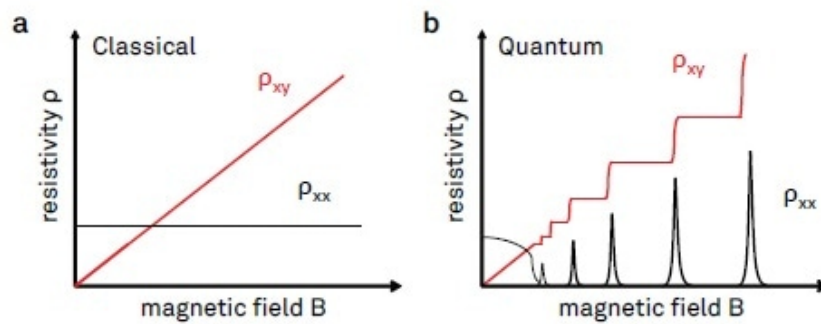


Figure 5.2: (a) The predicted values of the Classical Hall resistivities and (b) experimental results giving the quantized values for the resistivities.

Source: <https://www.azonano.com/article.aspx?ArticleID=5173>.

In order to understand why the resistivity acquires quantized values and what is the reason behind the plateaux present in figure 5.2, we need to study the quantum Hall effect. We start with the Landau problem, which describes a quantum formulation of an electron in a strong magnetic field.

## 5.2 THE LANDAU PROBLEM

Before introducing the Hamiltonian of the system there are two main points that worth to be highlighted. First, the electron occupies a finite space in our sample, as it describes a circular trajectory in the presence of a magnetic field [35]. Second, the electrons respect the Pauli Principle and so for the case of a many electron system, each electron occupy their own space in the sample. These two arguments suggest that a degeneracy should exist in the ground state of the system, since the circular orbit of the electron can be anywhere in the sample.

We are considering charged electrons restricted to 2+1 dimensions in a low temperature limit and a strong magnetic field, in a way that all electrons are polarized in the same direction and we can treat them as spinless fermions. We focus in the isolated problem of one electron and later on we intend to generalize it to many non-interacting particles. The Hamiltonian for one particle is given by

$$H = \frac{1}{2m_e} [(p_1 - A_1)^2 + (p_2 - A_2)^2], \quad (5.3)$$

being  $p_i = -i\partial_i$  with  $i = 1, 2$  representing the spatial coordinates.  $\vec{A}$  is the electromagnetic vector potential. The Hamiltonian can also be written with its conjugated momenta as

$$\Pi_i = p_i - A_i, \quad (5.4)$$

$$H = \frac{1}{2m_e} (\Pi_1^2 + \Pi_2^2). \quad (5.5)$$

Before we find the spectrum of this Hamiltonian, it is interesting that we inspect what kind of symmetries are contained in it.

### 5.3 MAGNETIC TRANSLATION

The Hamiltonian (5.3) is not invariant by translations because the vector potential depends explicit on the position, even though we have just argued in the beginning of this chapter that the physical system itself has some kind of translation symmetry. We want to find a new operator  $\vec{k}$  that is the generator of translations in this system. A translation by an infinitesimal vector  $\vec{a}$  is implemented as

$$t(\vec{a})^\dagger \vec{r} t(\vec{a}) = \vec{r} + \vec{a}, \quad (5.6)$$

being  $\vec{r}$  the position vector operator and  $t(\vec{a})$  given by

$$t(\vec{a}) = e^{-i\vec{a}\cdot\vec{k}}. \quad (5.7)$$

The conjugate momentum itself is not suitable to be the generator, as it does not commute with the Hamiltonian. We need that

$$[k_i, H] = 0. \quad (5.8)$$

It is natural that  $k_i$  is related to the conjugate momentum operator, which is the usual translation generator. We can define  $k_i$  to be the conjugate momentum plus an extra term, so that  $k_i$

commutes with the Hamiltonian. With that in mind, we define  $k_i$  as

$$k_i \equiv p_i - A_i + B\epsilon_{ij}x_j, \quad (5.9)$$

such that  $\epsilon_{ij}$  is the two component Levi-Civita symbol and the magnetic field is  $B = \epsilon_{ij}\partial_i A_j$ . The commutation relations between the components are

$$[k_i, k_j] = iB\epsilon_{ij}. \quad (5.10)$$

The operator  $\vec{k}$  is called the generator of magnetic translations [37]. Both of its components commute with the Hamiltonian and therefore are constants of motion. This also means that the system is invariant under magnetic translations, just what we were looking for. The magnetic translation operator in equation (5.7) obey the magnetic algebra [9], that is, given two infinitesimal vectors  $\vec{a}$  and  $\vec{b}$ ,

$$\begin{aligned} t(\vec{a})t(\vec{b}) &= e^{-iB\epsilon_{ij}a_ib_j}t(\vec{b})t(\vec{a}) \\ &= e^{-iB(\vec{a}\times\vec{b})\cdot\hat{z}}t(\vec{b})t(\vec{a}). \end{aligned} \quad (5.11)$$

Consider a particle in a point  $P$  that performs a closed path by four consecutive translations, represented in figure 5.3. This transformation is implemented in a state  $|\psi\rangle$  as

$$t(\vec{a})t(\vec{b})t(-\vec{a})t(-\vec{b})|\psi\rangle = e^{-iB(\vec{a}\times\vec{b})\cdot\hat{z}}|\psi\rangle. \quad (5.12)$$

We would expect the particle to return to its original state, although this is not exactly what happens. The particle gains a phase where

$$-iB(\vec{a}\times\vec{b})\cdot\hat{z} = -iAB = -i\Phi, \quad (5.13)$$

being  $A$  the area enclosed by the particle path and  $\Phi$  the magnetic flux.

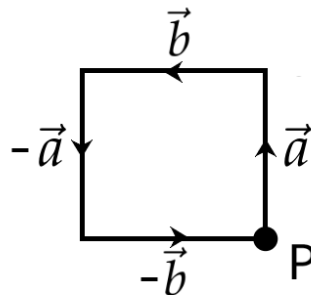


Figure 5.3: Four consecutive translations performed in one particle, starting at the point  $P$ .

Let us assume that the flux can be described as

$$\Phi = N_{\Phi}\Phi_0, \quad (5.14)$$

being  $N_{\Phi}$  the number of fluxes on the system and  $\Phi_0$  a fundamental flux called quantum flux, constituted by fundamental constants as the electron charge and the Planck constant [8]. We can assume that there are quantized numbers of flux quantum because every electron occupies the same area in the sample, which encloses one flux quantum [12]. That means that we cannot have fractional numbers of flux quantum since we have an integer number of electrons in the sample. In the natural units,

$$\Phi_0 \equiv 2\pi. \quad (5.15)$$

We can also define a magnetic length in the natural units system as

$$l_B^2 \equiv \frac{1}{B}. \quad (5.16)$$

The parameters defined in equations (5.15) and (5.16), as well as the magnetic translation invariance, will be useful when we deal with the degeneracy of the ground state. We now go back to the spectrum of the theory.

## 5.4 SPECTRUM

Going back to our Hamiltonian in the form

$$H = \frac{1}{2m_e}(\Pi_1^2 + \Pi_2^2), \quad (5.17)$$

we want to find its energy levels. Defining the creation and the annihilation operators as

$$a \equiv \frac{1}{\sqrt{2B}}(\Pi_1 - i\Pi_2), \quad a^\dagger \equiv \frac{1}{\sqrt{2B}}(\Pi_1 + i\Pi_2), \quad (5.18)$$

our Hamiltonian becomes

$$H = \omega_B(a^\dagger a + \frac{1}{2}), \quad (5.19)$$

where  $\omega_B = \frac{B}{m_e}$  is the cyclotron frequency. It is straightforward to see that the system given by the Hamiltonian (5.19) is similar to a harmonic oscillator, and thus the energy spectrum is

$$E_n = \omega_B(n + \frac{1}{2}), \quad (5.20)$$

with  $n$  representing the  $n$ th energy level, known as the  $n$ th Landau level. The levels are separated by an energy gap of  $\frac{B}{m_e}$ . Having analyzed the symmetry and the spectrum of the problem, we now move to the degeneracy problem.

#### 5.4.1 Degeneracy

In 2+1 dimensions the sample can be defined to be a rectangle with length  $L_1$  and  $L_2$  in two orthogonal directions  $\hat{e}_1$  and  $\hat{e}_2$ , respectively. We select the parameters of the operators from equation (5.11) to be  $\vec{a} = \frac{L_1}{N_\Phi} \hat{e}_1$  and  $\vec{b} = \frac{L_2}{N_\Phi} \hat{e}_2$ . These parameters are chosen so that the area in equation (5.13) is the total area of the sample  $A = L_1 L_2$ . In addition, we divided the vectors by the total number of fluxes  $N_\Phi$ , an integer number, so that after  $N_\Phi$  translations in each direction, we go back to the beginning of the sample, considering periodic boundary conditions. Replacing these parameters in equation (5.11) and using equations (5.13) and (5.15), we get

$$t(\vec{a})t(\vec{b}) = e^{-i\frac{2\pi}{N_\Phi}} t(\vec{b})t(\vec{a}). \quad (5.21)$$

Consider an eigenstate of the  $n$ th Landau level that is also an eigenstate of the translation operator  $t(\vec{a})$ , that is,

$$H\psi_{n,0} \equiv E_n\psi_{n,0}, \quad (5.22)$$

$$t(\vec{a})\psi_{n,0} \equiv e^{i\lambda_0}\psi_{n,0}. \quad (5.23)$$

Since both  $t(\vec{a})$  and  $t(\vec{b})$  commute with  $H$ , we can define a state  $\psi_{n,m}$  such as

$$H\psi_{n,m} \equiv Ht^m(\vec{b})\psi_{n,0} = t^m(\vec{b})H\psi_{n,0} = E_n\psi_{n,m}. \quad (5.24)$$

Based on equations (5.21) and (5.23), we can see that even though  $\psi_{n,m}$  and  $\psi_{n,0}$  have the same energy, they do not have the same eigenvalue for the operator  $t(\vec{a})$ . Explicitly,

$$t(\vec{a})\psi_{n,m} = e^{-2i\pi\frac{m}{N_\Phi} + i\lambda_0}\psi_{n,m}. \quad (5.25)$$

This means that given a Landau level  $n$  we have  $N_\Phi$  degenerate states, since  $m = 1, 2, \dots, N_\Phi$  [9]. Based on equations (5.13) and (5.14), the degeneracy of the system, given by  $N_\Phi$ , can also be written as

$$N_\Phi = \frac{AB}{2\pi}, \quad (5.26)$$

with  $A$  being the total area of the sample. This concludes the Landau Problem. The next step is to generalize our results to a many particle system.

## 5.5 GENERALIZATION OF THE LANDAU PROBLEM

In this section we intend to generalize the one particle problem, defined by the Landau problem, to a general system of  $N$  electrons. This ends up giving two different cases, the Integer Quantum Hall Effect (IQHE) and the Fractional Quantum Hall Effect (FQHE). We want to find the wave function for both cases and discuss some important properties of each.

### 5.5.1 Many Particles: The Integer Quantum Hall Effect

In order to generalize the Landau problem to a many particle system, we need to define what filling fraction means. The filling fraction is defined to be a parameter  $\nu$  that reflects the ratio between the number of electrons  $N$  and the number of possible states in a given Landau level  $N_\phi$  [35]. Explicitly,

$$\nu \equiv \frac{N}{N_\phi}. \quad (5.27)$$

Each Landau level has a degeneracy of  $N_\phi$  states in each level, as discussed in equation (5.26). We want to describe a system of many particles, but we need to be careful about the amount of electrons we intend to fill the system with. If we fill up all the states in the first Landau level, we get a filling fraction  $\nu = 1$ . If we fill up the first and second Landau levels,  $\nu = 2$ . These systems with integer values of filling fraction result in the integer quantum Hall effect. We will restrict our system to the filled ground state where  $\nu = 1$ , to see its properties. In this case,  $N = N_\phi$  and we have the same degeneracy discussed in the Landau problem. Also, we are considering non-interacting electrons.

Before defining a many particle function, we need the wave function for one particle. We follow reference [12] to find the eigenstates of the Landau Hamiltonian. First, we make a variable change to define our system with complex coordinates  $z = x + iy$  and  $\bar{z} = x - iy$ , so, in the symmetric gauge  $\vec{A} = (\frac{-By}{2}, \frac{Bx}{2})$ , the Hamiltonian in (5.3) becomes

$$H = -\frac{1}{m_e} \left[ (\partial_z - \frac{B\bar{z}}{4})(\partial_{\bar{z}} + \frac{Bz}{4}) + (\partial_{\bar{z}} + \frac{Bz}{4})(\partial_z - \frac{B\bar{z}}{4}) \right], \quad (5.28)$$

with  $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$  and  $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ . Using the canonical commutation relations between  $x_i$  and  $P_i = -i\partial_{x_i}$ , we can compute the creation and annihilation operators to be respectively

$$a^\dagger = \frac{1}{\sqrt{2B}}(\partial_z - \frac{B\bar{z}}{4}), \quad (5.29)$$

and

$$a = \frac{1}{\sqrt{2B}} \left( -\partial_{\bar{z}} - \frac{Bz}{4} \right), \quad (5.30)$$

such that

$$[a, a^\dagger] = 1. \quad (5.31)$$

With these operators, we can rewrite the Hamiltonian as

$$H = \omega_B \left( a^\dagger a + \frac{1}{2} \right). \quad (5.32)$$

The wave-function of the ground state of the system is defined as the state annihilated by the annihilation operator. With the explicit form of the annihilation operator in (5.30), we find the wave-function of the ground state to be

$$\psi_{0,m} = z^m e^{-\frac{|z|^2}{4l_B^2}}. \quad (5.33)$$

We can compute the wave-function of higher Landau levels by repetitively applying the creation operator, such that the eigenstate of the  $n$ th Landau level is

$$\psi_{n,m}(z, \bar{z}) \sim \left( \partial_z - \frac{\bar{z}}{4l_B^2} \right)^n z^m e^{-\frac{|z|^2}{4l_B^2}}. \quad (5.34)$$

Let us analyze the above equation. We see that the eigenstates depend on two quantum numbers, (i) the parameter  $n$  representing the  $n$ th Landau level, (ii) and the parameter  $m$  that represents the angular momentum. With the explicit form of the angular momentum operator

$$L_z = z\partial_z - \bar{z}\partial_{\bar{z}}, \quad (5.35)$$

we can see that the term  $z^m$  in equation (5.34) represents the eigenstates of the angular momentum operator with  $m$  as their eigenvalues, which supports our previous comment. This shows that the system has a degeneracy, such that, for a given energy that only depends on  $n$  (equation (5.20)), there are  $m$  eigenstates with different angular momentum. Also, the  $\sim$  sign in (5.34) means that we do not intend to normalize the function.

We can construct a many-particle free fermion ground state function through an anti-symmetric function, which in this case has the form of the Slater determinant [9], given

by

$$\det[\psi_i(x_j)] = \begin{vmatrix} \psi_1(x_1) & \psi_1(x_2) & \dots \\ \vdots & \ddots & \\ \psi_N(x_1) & \dots & \psi_N(x_N) \end{vmatrix}, \quad (5.36)$$

where  $\psi_i(x_j)$  is the wave function of the  $i$ th particle in the  $x_j$  position. Every particle must be at a different position because of the Pauli Principle. As we are computing the ground state function, we can replace equation (5.34) for  $n = 0$  into the matrix and find the many particle ground state function to be

$$\psi(z_1, z_2, z_3, \dots, z_N) = \begin{vmatrix} z_1^0 & z_2^0 & \dots \\ \vdots & \ddots & \\ z_1^{N-1} & \dots & z_N^{N-1} \end{vmatrix} e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}.$$

Let us analyze this wave function. Each single-particle state is represented by a column with a fixed coordinate  $z_i$ . Every state can have different angular momentum for the same energy value, which gives the different powers of  $z_i$ . The determinant also introduces the antisymmetric property in a very direct manner, because when we exchange two particles, two columns of the determinant are also exchanged, resulting in a minus sign. We can rewrite this equation as

$$\psi(z_1, z_2, z_3, \dots, z_N) = \prod_{i < j} (z_i - z_j) e^{-\sum_i \frac{|z_i|^2}{4l_B^2}} \quad (5.37)$$

It is important to realize that equation (5.37) respects the fact that two particles cannot be at the same location, because if  $z_i = z_j$ , the function vanishes. We call the many particle system in a strong magnetic field a Hall fluid.

There is a important property of the Hall system when the filling fraction acquires integer values. Given an integer filling fraction  $\nu$ , there are  $\nu$  Landau levels entirely filled. If we wanted to add a new electron to this system, it would have to go to the next Landau level represented by  $(\nu + 1)$ , which would cost a great amount of energy to overcome the gap between the levels. This leads to the incompressibility of the Hall fluid and the system behaves as an insulator [13]. Any attempt to compress the fluid would imply in one or more electrons jumping to the next Landau level, which is separated from the previous one by an energy of  $\omega_B$ , according to equation (5.20).

We are able to answer the first question proposed in section 5.1. Suppose that a small electric field is applied on the quantum Hall system, just as in figure 5.1. Starting with the Lorentz force and knowing that for  $\nu$  filled Landau levels the total charge is  $Q = \nu N_\phi$ , we

find the Hall quantum resistivity and the longitudinal quantum resistivity to be respectively

$$\rho_{xy} = \frac{2\pi}{\nu} \quad (5.38)$$

$$\rho_{xx} = 0. \quad (5.39)$$

The explicit calculations can be found in references [8, 9]. We can also relate these results to their respective conductivities

$$\rho_{xx} = 0 \rightarrow \sigma_{xx} = 0, \quad (5.40)$$

$$\rho_{xy} = \frac{2\pi}{\nu} \rightarrow \sigma_{xy} = \frac{\nu}{2\pi}. \quad (5.41)$$

The Hall resistivity is quantized by the values of  $\nu$  and the system behaves as an insulator when the filling fraction has integer values. But why there are plateaux in the graphic? The answer is related to disorder. Figures 5.4 and 5.5, associated to the effect of the impurities in the system, can be useful to understand this concept, as they represent the effect of impurities in the sample and in the energy spectrum, respectively.

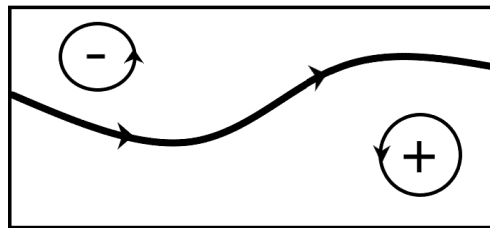


Figura 5.4: A schematic representation of the electron wave functions in the Hall system, in the presence of impurities. The minus sign represents the maximum of the potential and the minus sign the minimum.

Disorder can be seen as impurities, which may be treated as short ranged potentials placed randomly in the system [35], as shown in figure 5.4. All systems possess some kind of impurity in it. The presence of these potentials create localized states in the system, states where the electron wave function gets trapped. These states are represented by states orbiting the maximum and minimum of the potentials, in figure 5.4. The particle is not able to get out and thus do not contribute to the longitudinal electric current. On the other hand, there are extended states in the sample, states that allow the particle pass trough one side to the other by an equipotential, avoiding the potential wells and therefore carrying electric current. This state is represented by the line that crosses from one side to the other in figure 5.4. Only few states of the Hall fluid are extended [8].

The disorder also effects the Landau levels. We may look at figure 5.5. The presence of the potential causes the degeneracy of the Landau levels to decrease. The states at the upper and lower edges of the energy bands are the localized states, while the states at the

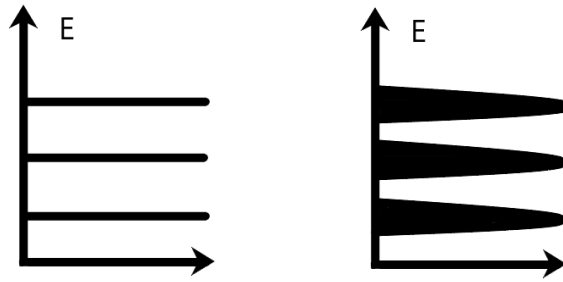


Figure 5.5: A graphic of the electron energy bands, being the left side the spectrum in the absence of disorder, and the right side in the presence of disorder.

middle are the extended. When an electron band is filled, the Hall conductivity stays constant until the Fermi energy reaches the point to fill the extended states of the next band, causing the conductance (as the resistivity) to jump to a non-vanishing value [8]. In other words, the localized states do not contribute to the filling fraction and neither to the change in  $\rho_{xy}$  since they do not add states to  $N_\phi$  in (5.27). This is the reason behind the plateaux in figure 5.2. In the next section, we will consider another case of the quantum Hall effect, the Fractional Quantum Hall Effect (FQHE), which arises when the filling fraction is smaller than one. That means that we are considering the first Landau level partially filled. We continue to follow the microscopic approach to study the FQHE.

### 5.5.2 Many Particles: The Fractional Quantum Hall Effect

So far we have seen some properties that appear for integer values of filling fraction in the Hall fluid considering non-interacting electrons. Experimentally, some of the same properties as the incompressibility and the insulator characteristic also appear for odd-denominator values of fraction filling fraction such as  $\nu = \frac{1}{3}, \frac{2}{3}, \dots$  [13]. This represents some kind of inconsistency in our previous example. Now that we have  $N_\phi$  possible states and consider a number of electrons  $N < N_\phi$ , the system has empty states, which would implicate in the compressibility of the system [9]. The answer is to include a new ingredient to our microscopic approach, interaction repulsion between electrons. The interaction plays a fundamental part in the fractional case [8].

The problem is that it is not easy to extract the physics of an interacting system from a microscopic point of view. We would have to propose a new Hamiltonian and compute all the properties again. Instead of going through this difficult path, Laughlin [10] came up with a new approach by proposing a variational wave function based on the non-interacting IQHE that suits the fractional problem. We start with the answer given by the Laughlin wave function and explain the properties and reasons behind it. For a filling fraction of

$$\nu = \frac{1}{m}, \quad m = 1, 3, 5, 7, \dots, \quad (5.42)$$

the Laughlin wave function is

$$\psi(z_1, z_2, z_3, \dots, z_N) = \prod_{i < j}^N (z_i - z_j)^m e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}, \quad (5.43)$$

such that  $m$  is the quantum number for the angular momentum and  $N$  is the number of particles. It is important that  $m$  only assumes odd numbers, so that our wave function is antisymmetric in the exchange of two particles, indicating that we are dealing with fermions. This wave function gives the same Hall resistivity proposed in equation (5.38), but with the filling fraction assuming values shown in equation (5.42).

Having the answer, we need to convince ourselves that it captures the essential physics of the quantum Hall effect. There are some properties of the state defined in (5.43) that must be checked in order to see whether it is really in agreement with our physical system. The first property to be analyzed is that if we consider  $m = 1$ , the equation (5.43) instantly recovers our previous result for the IQHE, seen in (5.37). This is a good start.

The next main point we want to check is whether this wave function actually has the desired filling fraction described by (5.42). Let us consider a single particle on  $z_1$ . For a  $N$  particles wave function, we would have  $m(N - 1)$  powers of  $z_1$  because of the product

$$\prod_{i < j}^N (z_1 - z_i)^m. \quad (5.44)$$

This means that, with the angular momentum operator given in (5.35), the maximum angular momentum of the particle on  $z_1$  is  $m(N - 1)$ . All the particles have the same energy, and so this state with the maximum angular momentum represents the larger area of the system, which corresponds to a total area  $A = 2\pi m N l_B^2$  of the droplet [8, 12]. We can use this result in equation (5.27) simultaneously with equations (5.26) and (5.16) to get

$$\nu = \frac{N}{N_\phi} = \frac{N}{BmNl_B^2} = \frac{N}{mN} = \frac{1}{m}. \quad (5.45)$$

Continuing to the other properties, if we consider two particles at the same place  $z_i = z_j$ , we get a zero of order  $m$  in equation (5.43), indicating a strong repulsive interaction between electrons [12]. Also because of the term  $(z_i - z_j)^m$  the wave function cannot be written as a product of each particle wave function. These two characteristics satisfy our comment about the existence of an interaction between the electrons.

Now that we have analyzed the Laughlin wave function and seen why it fits to our problem, we can go back to the question that started this section. Why is the fractional case still incompressible? Intuitively it is related to the presence of the Coulomb repulsive interaction between the electrons. Although the particles have a possibility of  $\frac{BA}{2\pi}$  states, they also have to

respect the repulsion between them, so they choose a spot and must be careful not to crash with other electrons. This is only possible for certain values of  $\nu$  [13].

The Laughlin wave function has an overlap of 99% between exact wave functions of small cluster of electrons and interaction pair potentials [9], which is extraordinary since its only parameter is the total angular momentum  $m$ . However, the Laughlin wave function may be seen as a approximate solution for the quantum Hall effect, since it is not a solution of the Schrödinger equation. Haldane [2] constructed Hamiltonians that actually had the Laughlin wave function as their exact solution. Now we departure from the microscopic point of view to a different approach. We want to understand the quantum Hall effect using an effective field theory in the context of quantum field theory, based on general properties of the Hall fluid.

## 5.6 EFFECTIVE FIELD THEORY

In the previous sections, we discussed the physical properties of the Hall fluid based on the microscopic degrees of freedom present in the system. The system may now be seen from a macroscopic point of view. We can obtain an effective action for the Hall fluid based on some general properties, resulting in the same quantized values for the Hall conductivity. What are the desired properties that our theory must have? (i) The system is restricted to a 2+1 spacetime dimensions; (ii) the electromagnetic current  $J^\mu$  is conserved; (iii) we want the field theory to be described by a local effective Lagrangian [13, 8]; (iv) we are interesting in a low-energy and long distances physics; and last (v) parity and time reversal symmetries are broken by the external magnetic field. Let us look more closely to each one of these properties.

The first property (i) just indicates that the particles are restricted to move in a two-dimensional plane. The second property (ii) is a standard result since we are not adding or removing electrical charge from the system. Mathematically this means that

$$\partial_\mu J^\mu = 0. \quad (5.46)$$

As we are in 2+1 spacetime dimensions, we can write the conserved current as a curl of a vector potential  $A_\mu = (A_0, A_1, A_3)$

$$J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda, \quad (5.47)$$

where  $\epsilon^{\mu\nu\lambda}$  is the Levi-Civita index and  $\frac{1}{2\pi}$  a normalization factor. Note that the current is unchanged by the gauge transformation  $A_\lambda \rightarrow A_\lambda - \partial_\lambda \Lambda$ , which means that  $A$  is a gauge potential.

Let us take a look at the next arguments (iii) and (iv). We want an effective theory that is not affected by the microscopic properties of the system, meaning that we are at long distances physics and low energies. Also, considering a low-energy limit, there are no degrees of freedom that can affect the physics when the system is perturbed [8]. The theory is

defined for the ground state of the Hall system, where there is an energy cost of the order of the gap to reach the first excited state. A perturbation smaller than the gap would not cause any particle to jump to the next state.

The last statement (v) is related to the magnetic field of the system that forces the electrons to move either clockwise or anticlockwise, breaking parity and time reversal. Having these general properties in mind, we want to find an effective action for low energies. We will not start with a Hamiltonian for the system, instead we want to define possible terms for the effective action that satisfies all the requirements above. Let us start with the integer quantum Hall effect.

### 5.6.1 IQHE: The Chern Simons Term

The effective action has the form  $S_{eff} = \int d^3x \dots$  and it has to be dimensionless. This means that the terms must have a mass dimension of 3. The term has to ensure that the system is gauge invariant, in accordance to arguments (iii) and (iv), and it has to break time reversal and parity symmetry. This excludes the possibility of having terms like  $A^\mu A_\mu$ , since it is not gauge invariant. The term that satisfies all these requirements is known as the Chern-Simons term, first introduced in reference [11]. The action is

$$S_{CS}[A] = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \quad (5.48)$$

In this equation the index  $CS$  indicates the Chern-Simons action and  $k$  is a constant that contains the physics of the magnetic field and other characteristics of the system [12].  $A$  is an external electromagnetic field applied on the system, enabling the Hall conductivity to be measured. It has nothing to do with the magnetic field that is now seen as a constant parameter.

There are some properties referent to the Chern-Simons action. It needs to break parity, time reversal and be gauge invariant. The transformations must keep the Maxwell's equations invariant. Parity is defined as

$$\begin{aligned} x_0 &\rightarrow x_0, \quad x_1 \rightarrow -x_1, \quad x_2 \rightarrow x_2, \\ A_0 &\rightarrow A_0, \quad A_1 \rightarrow -A_1, \quad A_2 \rightarrow A_2. \end{aligned} \quad (5.49)$$

Even though the integration factor is invariant under this transformation the integrand is not. Moving to time reversal, the transformation is

$$\begin{aligned} x_0 &\rightarrow -x_0, \quad x_1 \rightarrow x_1, \quad x_2 \rightarrow x_2, \\ A_0 &\rightarrow A_0, \quad A_1 \rightarrow -A_1, \quad A_2 \rightarrow -A_2. \end{aligned} \quad (5.50)$$

The action is also not invariant under this transformation. The last step is to analyze if the action

is gauge invariant. Under the gauge transformation  $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ , equation (5.48) becomes

$$S_{CS} \rightarrow S_{CS} + \int d^3x \partial_\mu (\Lambda \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda). \quad (5.51)$$

Usually, if the system is compact or infinite, we would just say that the total derivative term is a border term after the integration and send it to zero, but we have to be very careful here. The Hall system can be defined in a finite space and so nothing guarantees that the border term vanishes. We will take care of this issue, but for now we consider the space to be compact or infinite, so we do not have to worry about it.

The Chern-Simons term alone represents the integer quantum Hall problem. We could add the Maxwell's term  $F^{\mu\nu} F_{\mu\nu}$  to the action, but it is not relevant to the long distance physics since it has one more power of derivative than the Chern-Simons term [38]. To sustain our argument that there is no need of extra terms, we can compute the Hall conductivity out of equation (5.48) following some few steps. The electric current that arises from the action is

$$J_i = \frac{\delta S_{sc}}{\delta A_i} = -\frac{k}{2\pi} \epsilon_{ij} E_j, \quad (5.52)$$

with  $i, j = 1, 2$ . We can read off the conductivity of equation (5.52) as being the multiplication factor on the right side [12]. This means that the Chern-Simons action provides a conductivity

$$\sigma_{xy} = \frac{k}{2\pi}. \quad (5.53)$$

Imposing that  $k = \nu$  we get the Hall conductivity just as in equation (5.41). But what is the reason for us imposing that? Why can we do that? The way to prove that this is the right answer is related to the gauge invariant that we discussed previously.

Instead of imposing that that Chern-Simons term be gauge invariant, all we need is the partition function to be gauge invariant. This will impact in the restriction needed for  $k$ . The equivalence between the path integral used in QFT and the partition function in statistical mechanics can be found in appendix A. We now analyze the gauge invariance.

We use the Euclidean time for this approach and consider it to be a compact finite parameter, that is,  $\tau = \tau + \beta$ . Although it is strange to treat time as a cyclic parameter, it has a natural interpretation at a finite temperature. Arguments to sustain this property can be found in appendix A. Instead of considering that the parameter  $\Lambda(\tau, x)$  of the gauge transformation has to be single valued, we want it to assume values that leave the gauge transformation to be single valued. Under a gauge transformation, the electron field transforms as  $e^{ie\Lambda}$  [21]. Since we are in a compact geometry for the time coordinate, we need  $\Lambda(\tau, x)$  to restrict only this coordinate so that  $\Lambda(\tau)$  and  $\Lambda(\tau + \beta)$  represent the same transformation. With the definition

$$\Lambda \equiv \frac{2\pi\tau}{\beta}, \quad (5.54)$$

the transformation is single valued. This is called a large gauge transformation. Under this structure, the time-like component of the gauge field transforms as

$$A_0 \rightarrow A_0 + \frac{2\pi}{\beta}. \quad (5.55)$$

Before moving to the transformation of the Chern-Simons action, we need to make a last consideration. Instead of defining the Hall problem in a  $R^2$  surface as did before, we define it to be the superficial area in a  $S^2$  sphere with a monopole magnetic field inside [8]. With this geometry we can compute the magnetic flux

$$\frac{1}{2\pi} \int_{S^2} (\partial_1 A_2 - \partial_2 A_1) = 1. \quad (5.56)$$

This result comes from Dirac quantization condition in natural units [39]. Any magnetic flux cannot be smaller than the quantity on the right side, which is related to one electron charge in non-natural units. Although it sounds experimentally impossible, the consideration of a magnetic monopole does not disrespect the Maxwell's equations. Theoretically it is possible and so there is nothing wrong in our geometric approach to the problem. We shall analyze the gauge transformation on the action.

We fix  $A_0 = a$ , being  $a$  a constant. Expanding the components in (5.48) using the geometry defined above,

$$\begin{aligned} S_{CS} &= \frac{k}{4\pi} \int_{S^2} d^3x [A_0(\partial_1 A_2 - \partial_2 A_1) + A_1(\partial_2 A_0 - \partial_0 A_2) + A_2(\partial_0 A_1 - \partial_1 A_0)] \\ &= \frac{k}{4\pi} \int_{S^2} d^3x [A_0(\partial_1 A_2 - \partial_2 A_1) + A_1 \partial_2 A_0 + A_2 \partial_1 A_0] \\ &= \frac{k}{2\pi} \int_{S^2} d^3x [A_0(\partial_1 A_2 - \partial_2 A_1)] \\ &= \beta a k. \end{aligned} \quad (5.57)$$

In the first step of the calculation, we considered terms with time derivatives equal to zero. Looking at equation (5.56), we can see that the spatial components of the gauge field do not have any time dependence, since we did not integrate on the time and the right side does not have a time dependence. The next step was to integrate by parts the remaining terms disregarding total derivatives terms, which is possible because of the geometry of our system, followed by the substitution of the flux defined in equation (5.56).

Using equation (5.55), it is quite straightforward to see that, under the gauge transformation, the action becomes

$$S_{CS} \rightarrow S_{CS} + 2\pi k. \quad (5.58)$$

Replacing this result in the quantum partition function

$$Z[A] = \int D(\text{fields}) e^{iS_{eff}(\text{fields}, A)} = e^{iS_{CS}(A)}, \quad (5.59)$$

we can see that if  $k$  is proportional to integer numbers  $\mathcal{Z}$ ,

$$k = \nu \in \mathcal{Z}, \quad (5.60)$$

the partition function is gauge invariant. This is exactly the result we wanted in equation (5.53). Now that we have proven that the Chern-Simons action give us the same results for the IQHE, we move on to the FQHE.

### 5.6.2 Effective Theory for Laughlin States: FQHE

In the microscopic approach, the FQHE arises when interaction between electrons is taken into account. This resulted in fractional values for the filling fraction. In the effective theory, we have just proved that  $\nu$  must have integer values for the Hall fluid, so how should we approach this problem? The trick is that, besides what we have commented in the last section, there are in fact degrees of freedom that are gapped and still can affect the physics of the system at low-energy scales. They are called topological degrees of freedom. For now we get satisfied with this argument to say that it is possible to have fractional values for  $\nu$ . We discuss the topological nature of the Chern-Simons action in appendix B.

The topological degrees of freedom are implemented by an emergent gauge field  $a_\mu$  [14], which arises from the general behavior of the electrons in the Hall system. It contains information about the interaction between the particles and their dynamics. It is not whatsoever related to the usual electromagnetic potential. Our goal is the same as it was before, to write an effective action for the Hall effect described by Laughlin states. The partition function is given by

$$Z[A] = \int Da e^{iS_{eff}(a, A)}. \quad (5.61)$$

In order to write down  $S_{eff}(a, A)$ , there is the need to couple  $A_\mu$  and  $a_\mu$ .  $A_\mu$  has to couple with the conserved current, which can be written as in equation (5.47) plus some constant factors, that is

$$J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda. \quad (5.62)$$

The  $\frac{1}{2\pi}$  factor is needed to respect the Dirac quantization so the current gives a charge  $\int J^0 = 1$

[8]. Then the effective action is

$$S_{eff}(a, A) = \int d^3x \mathcal{L}_{eff} = \int d^3x \left[ -\frac{m}{4\pi}(a_\mu \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda) + \frac{1}{2\pi}(A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda) + \dots \right]. \quad (5.63)$$

The first term of the action has the same structure as the CS defined in equation (5.48) and so we can use the same arguments to affirm that  $m \in \mathcal{Z}$ . This term can also be called a Chern-Simons term. The second term is a current term coupled with  $A$ . The dots represents the Maxwell's term (which is not relevant) and the original Chern-Simons term for the field  $A$ . We could add the Chern-Simons term given in equation (5.48), but we have already seen that the result of this term is to give an integer contribution to the filling fraction. As the Laughlin states define the ground state of our theory, where the filling fraction is lesser than one, we can set this term to zero by making  $k = 0$ .

In the previous section the Hall conductance was found to be the current related to the  $A\partial A$  term. We need to rearrange the action in (5.63) to obtain the same structure. In order to do so, we integrate  $a_\mu$  out using its equation of motion, that is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial a_\mu} &= \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu a_\nu)}, \\ \partial_\nu a_\lambda - \partial_\lambda a_\nu &= \frac{1}{m} (\partial_\nu A_\lambda - \partial_\lambda A_\nu). \end{aligned} \quad (5.64)$$

Replacing this result into equation (5.63) we have

$$S_{eff} = \int d^3x \frac{1}{4\pi m} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \quad (5.65)$$

Following the same procedure done in equations (5.52) and (5.53), we get that

$$\sigma_{xy} = \frac{1}{2\pi m} = \frac{1}{2\pi} \nu. \quad (5.66)$$

The derivation of the fractional values of the filling fraction done above is not entirely accurate. Yes, we found the right Hall conductance for the Laughlin states, but we integrated out  $a_\mu$ , which means we cannot study the topological degrees of freedom. Also, equation (5.64) does not respect the Dirac quantization. If we attach one unit of flux to  $A$ , the left side will break the minimum quantization. On the other hand, if we do not integrate equation (5.64) over the space, there is nothing wrong with it, or if we considered a plane instead of a  $S^2$  manifold, the Dirac quantization would still be valid. Local quantities, such as the conductance, are not affected by the topology of the system, so we can consider a  $S^2$  manifold and that is why the result in equation (5.66) is valid. Therefore, we can consider the integration of  $a_\mu$  as a tool to compute the Hall conductance, but we shall not integrate this field if we want to analyze topological properties of the system.

Instead of working with the action in (5.65), we should work with the original

action in (5.63) to study the topological features of the system. The effect of the Chern-Simons term in (5.63) is to attach a flux quanta of  $\phi = \frac{1}{m}$  to each charged particle. This will give birth to quasi-particles and quasi-holes [9, 35, 12].

### 5.6.3 Quasi-Holes and Quasi-Particles

The effect of the many-body interaction of the system results in new excitations called anyons. Instead of electrons we have quasi-holes and quasi-particles, defined as excitations that couple with the gauge potential [13]. They have fractional statistics and charge [9]. We need to introduce their respective operators to the action. Defining a new gauge field  $\tilde{a}$ , we can define a current  $\tilde{J}$  to represent these excitations, given by

$$\tilde{J}^\mu \equiv \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \tilde{a}_\lambda. \quad (5.67)$$

The current couples to the internal field  $a$ , and so equation (5.63) becomes

$$\begin{aligned} S_{eff} &= \int d^3x \left[ -\frac{m}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda + q a_\mu \tilde{J}^\mu \right] \\ &= \int d^3x \left[ -\frac{m}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu (A_\lambda + q \tilde{a}_\lambda), \right] \end{aligned} \quad (5.68)$$

such that  $q = 1$  represents the physics of quasi-holes and  $q = -1$  the quasi-particles. In order to actually know the charge of this excitations, we must couple the current with the electromagnetic field  $A_\mu$ . For that we use the equation of motion for  $a$

$$(\partial_\mu a_\nu - \partial_\nu a_\mu) = \frac{1}{m} [(\partial_\mu A_\nu - \partial_\nu A_\mu) + (q \partial_\mu \tilde{a}_\nu - q \partial_\nu \tilde{a}_\mu)]. \quad (5.69)$$

Replacing into the action, we have

$$\begin{aligned} S_{eff} &= \frac{1}{4\pi m} \int d^3x \epsilon^{\mu\nu\lambda} (A_\mu + q \tilde{a}_\mu) \partial_\nu (A_\lambda + q \tilde{a}_\lambda) \\ &= \int d^3x \left[ \frac{1}{4m\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{q}{2\pi m} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu \tilde{a}_\lambda + \frac{q^2}{4m\pi} \epsilon^{\mu\nu\lambda} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda \right] \\ &= \int d^3x \left[ \frac{1}{4m\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{q}{m} A_\mu \tilde{J}^\mu + \frac{q^2}{4m\pi} \epsilon^{\mu\nu\lambda} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda. \right] \end{aligned} \quad (5.70)$$

As discussed, the current defined with the factor  $\frac{1}{2\pi}$  represents one electron charge  $e$ . Noting that we got the same result multiplied by  $\frac{q}{m}$ , we can safely say that the excitations have charge of  $Q = \frac{e}{m}$  for quasi-holes and  $Q = -\frac{e}{m}$  for quasi-particles. Even though the flux attached to the equation of motion (5.64) may not respect the Dirac quantization, we can multiply it by  $m$  and think as  $m$  quasi-holes (or particles) in a closed manifold moving independently of each other and possessing a total charge of  $e$ , respecting the Dirac quantization [8]. Therefore, these excitations are topological defects that have fractional charge and they

must be considered as a many-particle state, since we cannot build an elementary excitation of one quasi-particle or hole. As well as their charge, the quasi-holes and quasi-particles have fractional statistics [12, 13, 9].

### 5.6.4 Edge States

In section 5.6.1 we considered the Hall system to be in a manifold without boundaries. Here, we want to analyze the system when the quantum Hall fluid is confined in a manifold with boundaries, which results in gapless modes propagating on the edge of the sample [9]. They can be studied with both the wave function and the effective approaches, yet we are interested in the edge modes present in the effective description of the Laughlin states, as they will be important in the next chapter.

The effective action that describes the Laughlin states is

$$S_{CS}[a] = \frac{m}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho. \quad (5.71)$$

This is the same action we had in equation (5.63) but in the absence of the external electromagnetic field. Our goal is to analyze how the edge modes arise from the Chern-Simons action when we consider the Hall fluid in a region with boundaries. A practical choice is to define the Hall fluid in the plane  $(x, y > 0)$ , so that the boundary is the  $x$  axis. The presence of the boundary implies that the action is not gauge invariant for arbitrary  $\Lambda$ , as we have seen in equation (5.51). We need to evaluate what restrictions are needed for the gauge transformation so the action remains gauge invariant.

Let us consider the gauge transformation  $a_\mu \rightarrow a_\mu + \partial_\mu \Lambda$ , which results in a variation of the action as

$$S_{CS}[a] \rightarrow S_{CS}[a] + \frac{m}{4\pi} \int_{y=0} dx dt \Lambda (\partial_t a_x - \partial_x a_t). \quad (5.72)$$

We now impose that  $\Lambda(y=0) = 0$  so that the Chern-Simons action becomes gauge invariant again<sup>2</sup>. However, this implies that we no longer have a gauge transformation for  $y=0$ . The gauge transformation kills degrees of freedom and, considering it to vanish at the boundary, we are left with dynamical degrees of freedom living at the edge. We then need to derive an action that describes these degrees of freedom for  $y=0$ .

One way to study the edge degrees of freedom is to choose initial boundary conditions for the CS action and consider its respective equations of motion as a constraint [8]. Before choosing this condition, let us derive the equations of motion for the Chern-Simons action with boundary, so we can analyze which values would help us to describe the edge modes.

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<sup>2</sup>This is a more restrictive gauge invariance, since it is only valid when  $\Lambda(y=0) = 0$

Considering equation (5.71), a variation of the field varies the action as

$$\begin{aligned}
\delta S_{CS} &= \frac{m}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \left[ \frac{\partial(a_\mu \partial_\nu a_\rho)}{\partial(a_\lambda)} \delta a_\lambda + \frac{\partial(a_\mu \partial_\nu a_\rho)}{\partial(\partial_\nu a_\lambda)} \delta(\partial_\nu a_\lambda) \right] \\
&= \frac{m}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} [\delta a_\mu \partial_\nu a_\rho + a_\mu \partial_\nu \delta a_\rho] \\
&= \frac{m}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} [\delta a_\mu f_{\nu\rho} + \partial_\nu (a_\mu \delta a_\rho)].
\end{aligned} \tag{5.73}$$

For  $y \neq 0$ , the principle of least action leads to the equations of motion

$$\epsilon^{\mu\nu\rho} f_{\nu\rho} = 0, \tag{5.74}$$

which are the equations of motion for the Chern-Simons action in the absence of boundary. For  $y = 0$ , we need to set the last term of equation (5.73) to zero so that the system has the same equations of motion in the presence of the edge. Explicitly,

$$(\epsilon^{\mu\nu\rho} a_\mu \delta a_\rho)|_{y=0} = a_0 \delta a_x - a_x \delta a_0 = 0 \tag{5.75}$$

We can choose either  $a_x(y=0) = 0$  or  $a_t(y=0) = 0$ , as both lead to the desired equations of motion, but we can also choose a more general solution, given by

$$a_0 = v a_x \rightarrow (a_t - v a_x)|_{y=0} = 0. \tag{5.76}$$

In the above, the parameter  $v$  was introduced by hand and it represents the velocity of excitations on the edge. The bulk Chern-Simons action given by (5.71) has no knowledge of this velocity, resulting in the need to insert it in the boundary condition so we can determine the edge excitations dynamics [8].

With the boundary condition in equation (5.76), it seems a smart choice to extend this condition to the bulk, demanding that for any value of  $y$

$$(a_t - v a_x) = 0. \tag{5.77}$$

Before replacing equation (5.77) into (5.71), let us rewrite our gauge choice. The CS action is purely topological, which implies an invariance under general transformation of coordinates. We consider the coordinate changes

$$t' = t, \quad x' = x + vt, \quad y' = y, \tag{5.78}$$

with the field components transforming as

$$a'_{t'} = a_t - v a_x, \quad a'_{x'} = a_x, \quad a'_{y'} = a_y. \tag{5.79}$$

The action is invariant under these changes and can be written as

$$S_{CS} = \frac{m}{4\pi} \int d^3x' \epsilon^{\mu'\nu'\rho'} a'_{\mu'} \partial_{\nu'} a'_{\rho'}. \quad (5.80)$$

Also, the boundary condition (5.77) becomes

$$a'_{t'} = 0, \quad (5.81)$$

where we consider the equation of motion for  $a'_{t'}$  as a constraint given by

$$\frac{\delta S}{\delta a'_{t'}} = 0 \rightarrow f'_{i'j'} = 0. \quad (5.82)$$

The solutions for equation (5.82) are

$$a'_{i'} = \partial_{i'} \phi, \quad (5.83)$$

with  $\phi$  being a scalar field. Replacing equations (5.83) and (5.81) in (5.80), we have the edge action given by

$$\begin{aligned} S_{Edge} &= \frac{m}{4\pi} \int d^3x' \epsilon^{i'j'} a'_{i'} \partial_{t'} a'_{j'} \\ &= \frac{m}{4\pi} \int d^3x' \partial_{x'} \phi \partial_{t'} \partial_{y'} \phi - \partial_{y'} \phi \partial_{t'} \partial_{x'} \phi \\ &= \frac{m}{4\pi} \int d^3x' \partial_y (\partial_{x'} \phi \partial_{t'} \phi) - \partial_{y'} \partial_{x'} \phi \partial_{t'} \phi - \partial_{y'} \phi \partial_{t'} \partial_{x'} \phi \\ &= \frac{m}{4\pi} \int d^3x' \partial_y (\partial_{x'} \phi \partial_{t'} \phi) - \int dt' dy' (\partial_{y'} \phi \partial_{t'} \phi) \Big|_{x=-\infty}^{x=\infty} \\ &= \frac{m}{4\pi} \int_{y=0} d^2x' \partial_{t'} \phi \partial_{x'} \phi, \end{aligned} \quad (5.84)$$

where in the transition from the third to the fourth line, we did an integration by parts in the second term. In the original coordinates,

$$S_{Edge} = \frac{m}{4\pi} \int_{y=0} d^2x \partial_t \phi \partial_x \phi - v (\partial_x \phi)^2. \quad (5.85)$$

This action represents a boson theory in 1+1 dimensions living at the edge of the Hall system, and it is usually called the Floreanini-Jackiw action [40]. The equations of motion in the  $(t, x, y)$  coordinates are

$$\partial_t \partial_x \phi - v \partial_x^2 \phi = 0. \quad (5.86)$$

To observe the dynamics of the boson field, we define a new field as

$$\rho \equiv \frac{1}{2\pi} \frac{\partial \phi}{\partial x}, \quad (5.87)$$

so the equations of motion in (5.86) are

$$\partial_t \rho(x, t) - v \partial_x \rho(x, t) = 0. \quad (5.88)$$

This equation represents a chiral wave propagating at speed  $v$ , with solutions as  $\rho(x + vt)$ . This means that the edge excitations only propagate in one direction since solutions as  $\rho(x - vt)$  do not satisfy equation (5.88), which implies that there is a chiral boson living at the boundary.

The boson edge mode gives us a hint about the possibility of using the bosonization technique to study the quantum Hall effect since bosonic operators arose naturally in the fermionic theory. However, the chiral boson at the edge is given by a 1-dimensional theory, the same dimension we defined our bosonization technique but different from the 2-dimensional Hall fluid. We must somehow find a way to connect the 2-dimensional Hall system to 1-dimensional theories. In the following chapter, we study the quantum wires approach, which uses bosonization to describe the quantum Hall effect.

## 6 QUANTUM WIRES DESCRIPTION

As discussed in the last chapter, the effective C.S. theory describes important low-energy physical properties of the Hall fluid, for example, the existence of the edge states. However, this approach is not directly connected to the microscopic degrees of freedom of the model, since it was based on the general properties of the Hall system. In this chapter, we begin the study of the quantum wires description of the QHE, introduced by Kane [16], which consists of describing the Hall fluid as an array of coupled fermionic wires in 1+1 dimensions. We will show how this approach describes the quantum Hall effect physics using the bosonization technique and, later in chapter 7, we will also demonstrate how the quantum wires system may connect the microscopic degrees of freedom with the effective Chern-Simons theory.

The quantum wires can be seen as a microscopic approach since we define each wire with fermionic microscopical degrees of freedom. In addition, this model also describes in a very natural way the quasi-particles and the edge excitations [15]. Therefore, the quantum wires description is an intermediate between the microscopic and the effective description [18]. Our objective in this chapter is to study the quantum wires system and analyze for which limits it will describe the Laughlin states [17, 15, 16]. We start defining the uncoupled model.

### 6.1 NON-INTERACTING QUANTUM WIRES IN 1+1 DIMENSIONS

We begin with the general description of the wires model. The system consists of  $\mathcal{N}$  identical non-interacting wires in 1+1 dimensions in the presence of a perpendicular magnetic field, as displayed in figure 6.1. There are spinless fermions propagating along in a 1-dimensional lattice along each wire, and we take two consecutive wires to be separated by a distance of  $a$ .

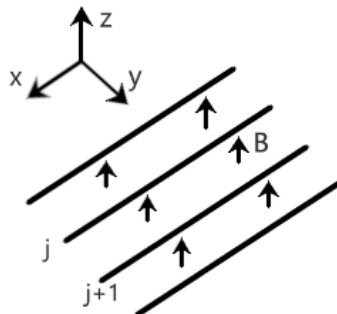


Figure 6.1: A schematic representation of the  $\mathcal{N}$  wires system in the presence of a perpendicular magnetic field. The wires are defined in the  $x$  coordinate and  $j$  is the wire index. Each wire  $j$  has right and left mover modes.

To describe spinless fermions in terms of Dirac fermions, we follow the same path as in section 4.3, but defining each wire to have the same electron density characterized by the Fermi momentum ( $K_F$ ), instead of the half-filled considered in section 4.3. We begin with the discrete Hamiltonian in the absence of a magnetic field, given by

$$H_0 = \sum_j^{\mathcal{N}} \sum_l^{\mathcal{M}} -\frac{1}{2m_e} \left( \psi_j^\dagger(l+1)\psi_j(l) + \psi_j^\dagger(l)\psi_j(l+1) \right) - \mu_0 \psi_j^\dagger(l)\psi_j(l), \quad (6.1)$$

where  $l$  is the site of the fermion,  $j$  is the wire index,  $m_e$  is the electron mass,  $\mu_0 = \frac{K_F^2}{2m_e}$  is a chemical potential that controls the filling of the system, and  $\mathcal{M}$  is the number of total sites in each wire. We can expand the fermion operators in the vicinity of the Fermi surface, where the energy is small. In the continuum limit, the fermion operator becomes

$$\psi_j(l) = \tilde{a}^{\frac{1}{2}} \left( e^{iK_{F,j}^R} \psi_{j,+}(x) + e^{iK_{F,j}^L} \psi_{j,-}(x) \right), \quad (6.2)$$

where  $\tilde{a}$  is the separation between the sites,  $K_{F,j}^{R/L} = \pm K_F$  are the right and left moving Fermi momenta, and  $\psi_{j,\pm}$  are the right and left Dirac fermions in the wire  $j$ .

Considering a low-energy limit and replacing equation (6.2) into (6.1), the Hamiltonian becomes

$$H_0 = \sum_j^{\mathcal{N}} v^F \int dx \left( \psi_{j,+}^\dagger(x)(-i\partial_x - K_F)\psi_{j,+}(x) - \psi_{j,-}^\dagger(x)(-i\partial_x + K_F)\psi_{j,-}(x) \right), \quad (6.3)$$

where  $v^F = \frac{K_F}{m_e}$  is the Fermi velocity and we take the separation between the wires to be small. Also, we have made the replacement  $\tilde{a} \sum_l \rightarrow \int dx$ . This Hamiltonian is continuum in the  $x$  coordinate, which allows us to implement the bosonization technique developed previously.

We shall now consider the presence of the magnetic field. The interaction with the perpendicular magnetic field shifts the right and left Fermi momenta on each wire by a constant [16]. With the momentum shift,  $K_{F,j}^R$  and  $K_{F,j}^L$  are given by

$$K_{F,j}^{R/L} = \pm K_F + bj, \quad (6.4)$$

with  $b = aB$  and  $a$  being the separation between the wires. In the presence of the magnetic field, the Hamiltonian becomes

$$H_0 = \sum_j^{\mathcal{N}} v^F \int dx \left( \psi_{j,+}^\dagger(x)(-i\partial_x - K_{F,j}^R)\psi_{j,+}(x) - \psi_{j,-}^\dagger(x)(-i\partial_x - K_{F,j}^L)\psi_{j,-}(x) \right). \quad (6.5)$$

Even though the Hamiltonian for the array of wires in (6.5) is a system in 2+1 dimensions, we are able to apply the bosonization relations studied in chapter 3 for each one-dimensional wire, which is an advantage of this approach. Before we proceed to the bosonization scheme, we

want to compute the filling fraction of the 2+1 dimensions system, which will be important to make the connection with the quantum Hall effect.

We can calculate the two-dimensional filling fraction using the one-dimensional electronic wire density. In the continuum limit, each wire has free electrons moving in the Fermi surface that are contained within the wire length. We parameterize this system as non-interacting particles moving inside an one-dimensional box [41]. The wave function for one particle trapped in an one-dimensional box is given by

$$\Psi(x) = C \sin(kx) = C \sin\left(\frac{n\pi x}{L}\right), \quad (6.6)$$

such that  $n = 0, 1, 2, \dots$ . The parameter  $C$  is a normalization constant and we take the length of the box to be  $L$ . The electron momentum  $k$  is restricted to assume values

$$k = \frac{n\pi}{L}. \quad (6.7)$$

For the wires system, we consider the one-dimensional problem for a particle with  $k = K_F$ , that is, the particle occupying the highest energy state at the Fermi surface. Since we are dealing with spinless fermions, there is only one electron in each occupied state. The integer value  $n$  is the number of occupied states below the Fermi surface and, since there is only one electron per occupied state,  $n$  characterizes the total number of electrons in each wire. Thus, the one-dimensional electron density  $\sigma_1$  is

$$\sigma_1 = \frac{\text{number of electrons in the wire}}{\text{length of the wire}} = \frac{n}{L} = \frac{K_F}{\pi}. \quad (6.8)$$

Using the separation  $a$  between two consecutive wires, the electron density of the  $\mathcal{N}$  wires system in two dimensions is

$$\begin{aligned} \sigma_2 &= \frac{\text{number of electrons in the 2-dimensional system}}{\text{area of the system}} \\ &= \frac{\mathcal{N}n}{A} = \frac{\mathcal{N}n}{\mathcal{N}La} = \frac{K_F}{\pi a}, \end{aligned} \quad (6.9)$$

where  $A$  is the area of the system and we have used equation (6.8) in the last equality. Also, recall that  $a$  is the separation between the wires. Finally, the filling fraction can be calculated as

$$\nu = \frac{\mathcal{N}n}{AB/2\pi} = \frac{2\pi\sigma_2}{B} = \frac{2K_F}{aB} = \frac{2K_F}{b}. \quad (6.10)$$

Based on (6.10), we can observe that the filling fraction of the system only depends on the Fermi momentum, the separation between the wires, and the intensity of the magnetic field. Restrictions that change one of these variables may control the filling. We now study the bosonization approach for the wires system, to apply it to the Hamiltonian in (6.5).

## 6.2 BOSONIZATION OF THE QUANTUM WIRES SYSTEM

The bosonization relation defined in equation (3.56) is suitable for all systems. Here, as we have an array of one-dimensional wires, we need to implement this relation in each wire separately while incorporating information about the entire system. It is natural that electron operators in different wires anti-commute with each other [17], which is not well defined in our bosonization relation. This section is devoted to rework the bosonization scheme to include anti-commutation and commutation rules between operators in different wires.

The anti-commutation relations between different wires are implemented by the Klein factors, defined by

$$\kappa_j \equiv e^{i\pi \sum_{i<j} N_i^R + N_i^L}, \quad (6.11)$$

with  $j$  being the wire index. The operators  $N_j^{R/L}$  are the right and left number operators, obtained through

$$N_j^p = \frac{p}{2\pi} \int dx \partial_x \phi_{p,j}, \quad (6.12)$$

with  $p = R, L = +, -^1$ . Implementing the Klein factors, the bosonization relation becomes

$$\psi_{j,p}(x) = \frac{\kappa_j}{\sqrt{2\pi\alpha}} e^{i(K_{F,j}^p x + \varphi_j + p\theta_j)}, \quad (6.13)$$

such that  $\alpha$  is a short-distance cutoff. The phase field  $\phi_j$  and the dual field  $\theta_j$  from the wire  $j$  are related to the bosonic right and left movers by

$$\varphi_j \equiv \frac{\phi_{j,+} + \phi_{j,-} + \pi N_j^L}{2}, \quad (6.14)$$

and

$$\theta_j \equiv \frac{\phi_{j,+} - \phi_{j,-} + \pi N_j^L}{2}. \quad (6.15)$$

For the array of wires, the commutation rules for the movers are

$$[\phi_{j,p}(x), \phi_{j',p'}(x')] = ip\pi \delta_{pp'} \delta_{jj'} \text{sign}(x - x'), \quad (6.16)$$

---

<sup>1</sup>We will use this notation for the indexes  $p$  and  $p'$  throughout this entire chapter, unless said otherwise.

and

$$\begin{aligned} [N_j^p, \phi_{j',p'}(x')] &= p \frac{1}{2\pi} \int dx [\partial_x \phi_{p,j}(x), \phi_{j',p'}(x')] \\ &= p \frac{1}{2\pi} \int dx \partial_x [\phi_{p,j}(x), \phi_{j',p'}(x')] = i\delta_{jj'} \delta_{pp'}. \end{aligned} \quad (6.17)$$

It is important to notice that the commutation relation in (6.16), for  $j = j'$  and  $p = p'$ , differ from a  $4\pi$  factor from the relation in (3.45), which means that each field  $\phi$  has absorbed a  $\sqrt{4\pi}$  factor. For operators in the same wire, the correlation functions defined in chapter 3 are still valid, as long as we remember to insert the  $4\pi$  factor for correlation functions of two fields.

We can also compute the commutation relations between  $\phi$  and  $\theta$  to be

$$\begin{aligned} [\theta_j(x), \varphi_{j'}(x')] &= \frac{1}{4} [\phi_{j,+}(x), \phi_{j',+}(x')] - \frac{1}{4} [\phi_{j,-}(x), \phi_{j',-}(x')] - \frac{\pi}{4} [\phi_{j,-}(x), N_{j'}^L] + \frac{\pi}{4} [N_{j'}^L, \phi_{j,-}(x)] \\ &= \frac{i\pi}{2} \delta_{jj'} \text{sign}(x - x') + \frac{i\pi}{2} \delta_{jj'} \\ &= i\pi \delta_{jj'} \Theta(x - x'), \end{aligned} \quad (6.18)$$

and

$$[\partial_x \theta_j(x), \varphi_{j'}(x')] = i\pi \delta_{jj'} \delta(x - x'), \quad (6.19)$$

with  $\Theta$  being the step function. Lastly, the trivial commutation rules are

$$[\theta_j(x), \theta_{j'}(x')] = [\varphi_j(x), \varphi_{j'}(x')] = 0, \quad (6.20)$$

and

$$[\kappa_j, \kappa_{j'}] = 0. \quad (6.21)$$

With the relations computed in this section, fermions in different wires anti-commute. Taking the indexes to be  $j > j'$ , we have

$$\begin{aligned} \{\psi_{j,p}, \psi_{j',p'}\} &= \frac{1}{2\pi\alpha} \left( k_j e^{i(\varphi_j + p\theta_j)} k_{j'} e^{i(\varphi_{j'} + p'\theta_{j'})} + k_{j'} e^{i(\varphi_{j'} + p'\theta_{j'})} k_j e^{i(\varphi_j + p\theta_j)} \right) \\ &= \frac{1}{2\pi\alpha} \left( k_j k_{j'} e^{i(\varphi_j + p\theta_j) + i(\varphi_{j'} + p'\theta_{j'})} + k_j k_{j'} e^{i(\varphi_{j'} + p'\theta_{j'}) + i(\varphi_j + p\theta_j) + i\pi(\frac{1-p'}{2}) + i\pi(\frac{1+p'}{2})} \right) = 0, \end{aligned} \quad (6.22)$$

where we did not consider the Fermi momentum terms since they commute with our variables. Having the bosonization relations reworked, we can apply them in the fermionic Hamiltonian in (6.5), to obtain the bosonic Hamiltonian.

### 6.3 THE FREE HAMILTONIAN

To replace the bosonization relation into (6.5), we need to compute two types of operators,  $\psi^\dagger(x)\psi(x)$  and  $\psi^\dagger(x)\partial_x\psi(x)$ . In general, there may be divergences when dealing with a product of two fermionic operators at the same space-time coordinate  $x$ . We must use the point splitting technique to analyze whether there are divergences or not when we consider the operators to be near the same point, so that the product is well defined [18]. Using this technique along with equation (6.13), it follows that

$$\psi_{j,p}^\dagger(x)\psi_{j,p}(x) \equiv \lim_{\epsilon \rightarrow 0} \left( \psi_{j,p}^\dagger(x+\epsilon)\psi_{j,p}(x) - \left\langle \psi_{j,p}^\dagger(x+\epsilon)\psi_j(x) \right\rangle \right), \quad (6.23)$$

and for  $p = +$ ,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \psi_{j,+}^\dagger(x+\epsilon)\psi_{j,+}(x) &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi\alpha} \kappa_j^* e^{-i(K_{F,j}^R(x+\epsilon)+\varphi_j(x+\epsilon)+\theta_j(x+\epsilon))} \kappa_j e^{i(K_{F,j}^R x+\varphi_j(x)+\theta_j(x))} \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi\alpha} \kappa_j^* \kappa_j e^{-i(K_{F,j}^R(x+\epsilon)+\varphi_j(x+\epsilon)+\theta_j(x+\epsilon))} e^{i(K_{F,j}^R x+\varphi_j(x)+\theta_j(x))} \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi\alpha} : e^{-i(K_{F,j}^R \epsilon + \varphi_j(x+\epsilon) + \theta_j(x+\epsilon) - \varphi_j(x) - \theta_j(x))} : e^{4\pi G_+(\epsilon)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{i}{2\pi(\epsilon - i\alpha)} : e^{-i\epsilon(K_{F,j}^R + \partial_x \varphi_j(x) + \partial_x \theta_j(x))} : \\ &= \lim_{\epsilon \rightarrow 0} \frac{i}{2\pi(\epsilon - i\alpha)} : (1 - i\epsilon(K_{F,j}^R + \partial_x \varphi_j(x) + \partial_x \theta_j(x))) : \\ &= \lim_{\epsilon \rightarrow 0} \frac{i}{2\pi\epsilon} + \frac{1}{2\pi} : (K_{F,j}^R + \partial_x \varphi_j(x) + \partial_x \theta_j(x)) : . \end{aligned} \quad (6.24)$$

In the above calculation,  $\kappa_j^*$  refers to the complex conjugate of the Klein factor and we used equations (3.50) and (3.47) between lines two and three, remembering to insert the multiplying factor of  $4\pi$  to the correlation function. We also have let  $\alpha \rightarrow 0$ , since it is a convergence factor. Performing the same calculation for the left movers, the operators given by equation (6.23) are

$$\psi_{j,p}^\dagger(x)\psi_{j,p}(x) = \frac{p}{2\pi} (\partial_x(\varphi_j(x) + p\theta_j(x))). \quad (6.25)$$

We follow the same path for the second type of operator,

$$\begin{aligned} \psi_{j,p}^\dagger(x)\partial_x\psi_{j,p}(x) &= \lim_{\epsilon \rightarrow 0} \left( \psi_{j,p}^\dagger(x) \frac{\psi_{j,p}(x+\epsilon) - \psi_{j,p}(x-\epsilon)}{2\epsilon} + \right. \\ &\quad \left. - \left\langle \psi_{j,p}^\dagger(x) \frac{\psi_{j,p}(x+\epsilon) - \psi_{j,p}(x-\epsilon)}{2\epsilon} \right\rangle \right), \end{aligned} \quad (6.26)$$

and

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} \psi_{j,+}^\dagger(x) \frac{\psi_{j,+}(x+\epsilon) - \psi_{j,+}(x-\epsilon)}{2\epsilon} \\
&= \lim_{\epsilon \rightarrow 0} \frac{-i}{4\epsilon\pi(\epsilon - i\alpha)} : (1 - i\epsilon((K_{F,j}^R + \partial_x\varphi_j(x) + \partial_x\theta_j(x))) - \frac{\epsilon^2}{2}(K_{F,j}^R + \partial_x\varphi_j(x) + \partial_x\theta_j(x))^2) : \\
&- \frac{i}{4\epsilon\pi(\epsilon - i\alpha)} : (1 + i\epsilon((K_{F,j}^R + \partial_x\varphi_j(x) + \partial_x\theta_j(x))) - \frac{\epsilon^2}{2}(K_{F,j}^R + \partial_x\varphi_j(x) + \partial_x\theta_j(x))^2) : \\
&= \lim_{\epsilon \rightarrow 0} \frac{-i}{2\epsilon\pi\epsilon} + : \left( \frac{iK_{F,j}^R}{2\pi}(\partial_x(\varphi(x) + \theta(x))) + \frac{i}{4\pi}(\partial_x(\varphi_j + \theta_j))^2 \right) : . \tag{6.27}
\end{aligned}$$

Plugging the results for the right and left movers in equation (6.26), we have

$$\psi_{j,p}^\dagger(x)\partial_x\psi_{j,p}(x) = \frac{ipK_{F,j}^p}{2\pi}(\partial_x(\varphi(x) + p\theta(x))) + \frac{ip}{4\pi}(\partial_x(\varphi_j + p\theta_j))^2. \tag{6.28}$$

Finally, replacing equations (6.25) and (6.28) into the Hamiltonian in (6.5), we obtain the bosonized Hamiltonian

$$\begin{aligned}
H_0 &= \sum_j^{\mathcal{N}} v^F \int dx \left( \psi_{j,+}^\dagger(x)(-i\partial_x - K_{F,j}^R)\psi_{j,+}(x) - \psi_{j,-}^\dagger(x)(-i\partial_x - K_{F,j}^L)\psi_{j,-}(x) \right) \\
&= \frac{v^F}{2\pi} \sum_j^{\mathcal{N}} \int dx \left( (\partial_x\varphi_j)^2 + (\partial_x\theta_j)^2 \right). \tag{6.29}
\end{aligned}$$

For each wire  $j$ , the Hamiltonian represents free massless bosons in 1+1 dimensions. This Hamiltonian is the bosonized form of the non-interacting array of wires and, as there are only kinetic terms, there is no energy gap in the system. Our next step is to insert interactions between the wires, which will lead the system to a gapped phase [17, 16], such as the Hall fluid.

We can also compute the charge density operator of the system, which will be used as a parameter when we analyze the interactions. According to equation (3.28), the fermionic charge density operator in each wire is

$$J_j(x) = J_j^R + J_j^L = \psi_{j,+}^\dagger\psi_{j,+} + \psi_{j,-}^\dagger\psi_{j,-} = \frac{\partial_x\theta}{\pi}, \tag{6.30}$$

so the total charge density operator for the array of wires is

$$J(x) = \sum_j J_j(x). \tag{6.31}$$

These operators will be important when we demand that interactions in the system conserve charge. We want to add interactions between the wires and, after we define general cases, we intend to find specific operators that let the coupled wires system to represent the FQHE physics.

## 6.4 COUPLED WIRES SYSTEM

To couple the non-interacting wires, we consider two types of interactions: *i*) a forward interactions, that do not open a gap in the system, and *ii*) interwire (tunneling) interactions, which have the potential to gap the system. We will introduce general interactions of both types and then evaluate the filling fraction and the renormalization character of the interacting system.

### 6.4.1 Forward Scattering Interaction

We first consider a many-body forward interaction, in which the electrons scatter without changing their direction of motion [9]. This interaction may be written in terms of densities, such that the general form of the interaction is given by

$$\begin{aligned} H_{fs} &= \sum_{jk,pp'} \int dx \psi_{j,p}^\dagger \psi_{j,p} V_{pp'}^{jk} \psi_{k,p'}^\dagger \psi_{k,p'} \\ &= \sum_{jk} \int dx \begin{pmatrix} \partial_x \varphi_j & \partial_x \theta_j \end{pmatrix} \mathbf{U}_{jk} \begin{pmatrix} \partial_x \varphi_k \\ \partial_x \theta_k \end{pmatrix}, \end{aligned} \quad (6.32)$$

where  $p = +, -$  and  $p' = +, -$ ,  $j$  and  $k$  run from 0 to  $\mathcal{N}$ ,  $V_{pp'}^{jk}$  represents the coupling constants between the density operators, and  $\mathbf{U}_{jk}$  is a 2x2 matrix that describes the forward scattering interaction. These interactions may lead to the renormalization of  $\varphi$  and  $\theta$ , which affects the relevance of the tunneling operators that we will introduce in the next section [17]. As the forward scattering operator is invariant under  $\phi_{j,+/-} \rightarrow \phi_{j,+/-} + c$ , this operator conserves charge in each wire and does not generate an energy gap.

Plugging the interaction term into the system, the Hamiltonian becomes

$$H_{SLL} = H_0 + H_{fs} = \sum_{jk} \int dx \begin{pmatrix} \partial_x \varphi_j & \partial_x \theta_j \end{pmatrix} \mathbf{M}_{jk} \begin{pmatrix} \partial_x \varphi_k \\ \partial_x \theta_k \end{pmatrix}, \quad (6.33)$$

where  $\mathbf{M}_{jk} = \delta_{jk} \mathbf{I} v_F / 2\pi + \mathbf{U}_{jk}$ . This Hamiltonian describes a sliding Luttinger liquid [42], which justifies the *SLL* index. We now study the interwire operator, which we will also refer to as a tunneling operator.

### 6.4.2 Interwire Interaction

The second type of interaction is an interwire interaction that allows tunneling of electrons between neighboring wires, which then violates charge conservation of individual wires [18]. This interaction can open an energy gap and may lead to different phases. The

tunneling operator that generates the interwire interaction has a general form

$$\mathcal{O}_j^{m_q n_q}(x) = \prod_q (\psi_{j+q,+})^{s_q^R}(x) (\psi_{j+q,-})^{s_q^L}(x), \quad (6.34)$$

where  $s_q^R$  and  $s_q^L$  are integer numbers. When they assume negative values, we take the operator to be its Hermitian conjugated, for instance,  $(\psi_{j+q,+})^{-2} \equiv (\psi_{j+q,+}^\dagger)^2$ . This operator allows charge to be exchanged between different wires, but it still conserves the total charge of the system since we are not adding or removing electrons from the wires.

It is useful to parameterize the operator numbers as

$$s_q^R \equiv \frac{n_q + m_q}{2} \quad (6.35)$$

and

$$s_q^L \equiv \frac{n_q - m_q}{2}, \quad (6.36)$$

with  $n_q$  and  $m_q$  also being integer numbers. Since  $s_q^{R/L}$  must assume integer values,  $m_q$  must be the rest of  $n_q/2$ , that is,

$$m_q = n_q \bmod 2. \quad (6.37)$$

Replacing equations (6.13), (6.35) and (6.36) into equation (6.34), the operator in the bosonized form is

$$\mathcal{O}_j^{m_q n_q}(x) = C \prod_q k_{j+q}^{n_q} R^{m_q n_q} e^{i(n_q \varphi_{j+q}(x) + m_q \theta(j+p)(x))}, \quad (6.38)$$

with  $C$  being a normalization constant and  $R^{m_q n_q}$  being the net momentum given by

$$R^{m_q n_q} = e^{(ibqn_q + K_F m_q)x}, \quad (6.39)$$

recalling that  $b = aB$ . It is implicit that we must consider the point split technique to evaluate the tunneling operator in (6.34) for values of  $s_q^{R/L} > 1$ , so that there are no divergences. When we consider the bosonization relations and the operators to be at the same space-time coordinate, the point splitting will only lead to trivial terms and so we do not use the explicit technique such as in equations (6.23) and (6.26).

We must check for which conditions the interwire operator satisfies the symmetries present in (6.33). First, we want the operator to preserve momentum, that is, to be invariant under translations. For the operator to have this symmetry, given a transformation  $x \rightarrow x + \delta x$ , there cannot be additional contributions besides the field operators contributions.

Thus, we need that

$$\sum_q bqn_q + K_F m_q = 0, \quad (6.40)$$

where the sum in  $q$  comes from the product in (6.38).

The second requirement is that even though the charge on each wire is not conserved, as the operator does not preserve the electron number in each wire, the total charge of the system must be invariant. Thereby, under the transformation  $\psi \rightarrow e^{i\lambda}\psi$  and  $\psi^\dagger \rightarrow e^{-i\lambda}\psi^\dagger$ , we need the tunneling operator to be invariant, which implies that

$$\sum_q s_q^R + s_q^L = \sum_q n_q = 0. \quad (6.41)$$

Thus, the interwire operator must assume values that satisfy (6.40) and (6.41).

The interwire interaction leads to a new term in the Hamiltonian, given by

$$H_{inter} = \sum_j \int dx (v^{m_q n_q} \mathcal{O}_j^{m_q n_q}(x) + h.c.), \quad (6.42)$$

with  $v^{m_q n_q}$  being a coupling constant. The full Hamiltonian in the presence of forward scattering and interwire interaction is

$$H = H_0 + H_{fs} + H_{inter} = H_{SLL} + \sum_j \int dx (v^{m_q n_q} \mathcal{O}_j^{m_q n_q}(x) + h.c.). \quad (6.43)$$

We are now able to compute the filling fraction of the system to study how these interactions may lead to the fractional filling fraction of the Laughlin states. We also analyze whether these interaction operators are relevant or not after we apply the renormalization group, which will give us knowledge of how the interactions affect the system.

### 6.4.3 Filling Fraction and the Renormalization Group

With the restrictions in equations (6.40) and (6.41), we can compute the filling fraction of the system, given by equation (6.10), to be

$$\nu = 2 \frac{\sum_q q n_q}{\sum_q m_q}. \quad (6.44)$$

The filling fraction only depends on  $m_q$  and  $n_q$ , integer numbers present in the interwire interaction operator. This indicates that if we want to characterize a specific filling fraction, like the one present in the FQHE, it will depend on the form of the interwire operator.

We shall make a brief analysis about the renormalization character of (6.43).  $H_{SLL}$  represents a fixed point of the theory. We have already computed a bosonic action with

this kind of term in equation (4.41) (remember that  $\partial_x \theta = \Pi$ ) and saw that the kinetic terms are invariant under the RG. Thus,  $H_{SLL}$  is a fixed point, which may be unstable if the interwire interaction is relevant.

As stated, the forward scattering leads to the renormalization of the fields and can affect the RG of tunneling operators. For certain values of the forward interaction given by  $U_{jk}$ , the renormalization of the fields  $\phi$  and  $\theta$  can result in the irrelevance of the tunneling operator. Following equation (2.54), the beta function for the interwire operator is

$$\frac{dv^{m_q n_q}}{d\mu} = (2 - \Delta^{m_q n_q})v^{m_q n_q}, \quad (6.45)$$

where  $\Delta^{m_q n_q}$  is the scaling dimension of the interwire operator, affected by the forward scattering.

If  $\Delta^{m_q n_q} > 2$ , the operator is irrelevant and does not generate an energy gap, as the coupling constant decreases when we go to low energies. If  $\Delta^{m_q n_q} < 2$ , the operator is relevant and generates an energy gap at low-energies. As the forward scattering interaction does not open an energy gap, its specific form is not important for now. We just assume the interaction to be in such way that let  $\Delta^{m_q n_q} < 2$ , leading the interwire operator to be relevant.

Until this point, we were describing general interactions compatible with certain physical requirements of the coupled wires system. We now intend to analyze in which limit the coupled wires system may represent the Laughlin states. Since the forward interaction is not related to the filling fraction and does not open an energy gap, we remain considering it in a general form that leads the interwire operator to be relevant, but we shall choose a precise interwire operator that leads to the Laughlin states.

## 6.5 LAUGHLIN STATES

In chapter 5, we saw that the Laughlin states present the fractional quantum Hall effect physics and in the effective approach, these states describe topological excitations (quasi-particles) and edge modes. In this section, we want to find a limit in which the coupled wires system describes this same physics. We need to find the interwire interaction that drives the model to have these desired features. We introduce the tunneling operator that generates the fractional filling fraction and study if our coupled system describes the existence of bulk quasi-particles and the presence of edge states.

### 6.5.1 The Tunneling Operator

The tunneling operator that leads to the filling fraction of the fractional quantum Hall effect is

$$\begin{aligned} \mathcal{O}_{l=j+1/2}^{1/m} &\equiv \left( (\psi_{l+q,+})^{(\frac{m-1}{2})} (\psi_{l+q,-}^\dagger)^{(\frac{m+1}{2})} \right) \Big|_{q=\frac{1}{2}} \left( (\psi_{l+q,+})^{(\frac{m+1}{2})} (\psi_{l+q,-}^\dagger)^{(\frac{m-1}{2})} \right) \Big|_{q=-\frac{1}{2}} \\ &= (\psi_{j+1,+})^{(\frac{m-1}{2})} (\psi_{j+1,-}^\dagger)^{(\frac{m+1}{2})} (\psi_{j,+})^{(\frac{m+1}{2})} (\psi_{j,-}^\dagger)^{(\frac{m-1}{2})}. \end{aligned} \quad (6.46)$$

We are defining the operator on the link  $l = j + \frac{1}{2}$  with the product for  $q = \pm \frac{1}{2}$ , so we can observe that the operator acts on the wires  $j$  and  $j + 1$ . Also,  $m$  must assume integer odd values so the power of the operators are also integer numbers as stated in (6.35) and (6.36). Using equation (6.44), we can compute the filling fraction of the system to be

$$\nu = 2 \frac{\sum_q q n_q}{\sum_q m_q} = \frac{(\frac{1}{2} + \frac{1}{2})}{2m} = \frac{1}{m}, \quad (6.47)$$

with

$$m = 1, 3, 5, \dots \quad (6.48)$$

Being  $m$  an odd integer, equation (6.47) represents the filling fraction of the FQHE. This is the first clue that the coupled wires system in the presence of the tunneling operator  $\mathcal{O}_{l=j+1/2}^{1/m}$  represents the Laughlin states.

Before we find the bosonized form of the operator, we want to track its effect on the system. Analyzing equation (6.46) and knowing that dagger operators create particles,  $\mathcal{O}_{l=j+1/2}^{1/m}$  destroys  $(\frac{m-1}{2})$  right movers and creates  $(\frac{m+1}{2})$  left movers on the wire  $j$ , while it destroys  $(\frac{m+1}{2})$  right movers and creates  $(\frac{m-1}{2})$  left movers on the wire  $j + 1$ . The resulting effect is the backscattering of  $(\frac{m-1}{2})$  electrons on both wires  $j$  and  $j + 1$ , while one electron hops from the wire  $j$  to  $j + 1$ . The effect of the operator can be seen in figure 6.2, where we analyze the examples for  $m = 1, 3$ .

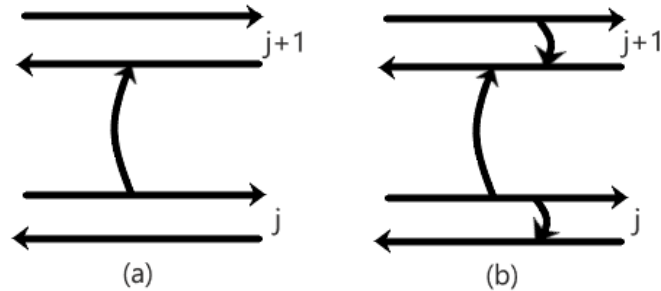


Figure 6.2: Schematic representation of the effect of the tunneling operator. The vertical arrows indicates an electron being destroyed in the wire  $j$  and being created in  $j + 1$ . We analyze the states for: (a)  $m = 1$ , which gives  $\nu = 1$ , and (b)  $m = 3$ , which gives  $\nu = \frac{1}{3}$ .

We now apply the bosonization relations. Replacing equation (6.13) into (6.46), we have

$$\begin{aligned}
\mathcal{O}_{l=j+1/2}^{1/m} &= \frac{1}{(2\pi\alpha)^2} k_{j+1}^{(\frac{m-1}{2})} k_{j+1}^{*(\frac{m+1}{2})} k_j^{(\frac{m+1}{2})} k_j^{*(\frac{m-1}{2})} e^{i(\frac{m-1}{2})(K_{F,j+1}^R x + \varphi_{j+1} + \theta_{j+1})} e^{-i(\frac{m+1}{2})(K_{F,j+1}^L x + \varphi_{j+1} - \theta_{j+1})} \times \\
&\times e^{i(\frac{m+1}{2})(K_{F,j}^R x + \varphi_j + \theta_j)} e^{-i(\frac{m-1}{2})(K_{F,j}^L x + \varphi_j - \theta_j)} \\
&= \frac{1}{(2\pi\alpha)^2} e^{-i\pi(N_j^R + N_j^L)} e^{i(\frac{m-1}{2})(K_{F,j+1}^R x + \varphi_{j+1} + \theta_{j+1})} e^{-i(\frac{m+1}{2})(K_{F,j+1}^L x + \varphi_{j+1} - \theta_{j+1})} \times \\
&\times e^{i(\frac{m+1}{2})(K_{F,j}^R x + \varphi_j + \theta_j)} e^{-i(\frac{m-1}{2})(K_{F,j}^L x + \varphi_j - \theta_j)} \\
&= \frac{1}{(2\pi\alpha)^2} e^{-i\pi(N_j^R + N_j^L)} e^{i(\varphi_j - \varphi_{j+1} + m(\theta_j + \theta_{j+1}))} e^{\frac{m^2-1}{4}[\theta_{j+1}, \varphi_{j+1}]} e^{\frac{m^2-1}{4}[\theta_j, \varphi_j]} \\
&= \frac{1}{(2\pi\alpha)^2} e^{-i\pi(N_j^R + N_j^L)} e^{i(\varphi_j - \varphi_{j+1} + m(\theta_j + \theta_{j+1}))} e^{i\pi(\frac{m^2-1}{2})} \\
&= \frac{1}{(2\pi\alpha)^2} e^{i(\varphi_j - \varphi_{j+1} + m(\theta_j + \theta_{j+1}) - \pi(N_j^R + N_j^L)) + i\pi(\frac{m^2-1}{2}) + \frac{i\pi}{2}} \\
&= \frac{1}{(2\pi\alpha)^2} e^{i(\varphi_j - \varphi_{j+1} + m(\theta_j + \theta_{j+1}) - \pi(N_j^R + N_j^L) + \frac{\pi}{2})}. \tag{6.49}
\end{aligned}$$

To perform this calculation we used the BCH theorem in the second step and used the fact that  $m$  is an odd integer in the last step, so  $e^{i\pi(\frac{m^2-1}{2})} = 1$ . Also, the net-momentum term must vanish because of equation (6.40). Replacing the bosonized form of the operator into the Hamiltonian in (6.43) and taking  $v^{m_q n_q}$  to be real, we have

$$\begin{aligned}
H &= H_0 + H_{fs} + H_{inter} \\
&= H_{SLL} + \sum_j \int dx \frac{v^{1/m}}{\pi} \sin(\varphi_j - \varphi_{j+1} + m(\theta_j + \theta_{j+1}) - \pi(N_j^R + N_j^L)), \tag{6.50}
\end{aligned}$$

where  $v^{1/m}$  is a coupling constant that includes  $v^{m_q n_q}$  and other factors. We define the forward interactions so that the sine operator is relevant and the tunneling operator leads the system to a gapped phase. To see that this gapped phase is actually the Laughlin states, we need to analyze further its properties.

Instead of working with the Hamiltonian in (6.50), we introduce new operators that reveal the quasi-particles and the edge states physics more naturally in the system. We follow the appendix from reference [15] and define the new right and left movers as

$$\tilde{\phi}_{j,+} \equiv \frac{1+m}{2} \phi_{j,+} + \frac{1-m}{2} \phi_{j,-}, \quad \tilde{\phi}_{j,-} \equiv \frac{1-m}{2} \phi_{j,+} + \frac{1+m}{2} \phi_{j,-}, \tag{6.51}$$

such that their commutation rules are

$$\left[ \partial_x \tilde{\phi}_{j,p}(x), \tilde{\phi}_{j',p'}(x') \right] = 2i\pi m p \delta_{pp'} \delta_{jj'} \delta(x - x'). \tag{6.52}$$

We also define the new electron right and left movers as

$$\tilde{\psi}_{j,\pm} \equiv (\psi_{j,\mp}^\dagger)^{\frac{m-1}{2}} (\psi_{j,\pm})^{\frac{m+1}{2}} = \frac{k'_{j,\pm}}{2\pi\alpha} e^{i(K_{F,j}^{R/L} x + \tilde{\phi}_{j,\pm})}, \quad (6.53)$$

with  $k'_{j,\pm} = k_j e^{i\pi(\frac{1\pm m}{2})N_j^L}$ . The new operators were defined such that they have the same commutation rules as the original right and left movers, with a  $m$  multiplying factor. Hence, the system remains constituted by fermions.

We then construct the new phase and density fields to have the same structure as the original fields,

$$\tilde{\theta}_{l=j+1/2} = \frac{\tilde{\phi}_{j,+} - \tilde{\phi}_{j+1,-} + \pi\tilde{N}_j^\theta}{2}, \quad \tilde{\varphi}_{l=j+1/2} = \frac{\tilde{\phi}_{j,+} + \tilde{\phi}_{j+1,-} + \pi\tilde{N}_j^\varphi}{2}, \quad (6.54)$$

with

$$\tilde{N}_j^\theta \equiv \frac{m-1}{2}N_j^L + N_j^R + \frac{m-1}{2}N_{j+1}^L, \quad (6.55)$$

and

$$\tilde{N}_j^\varphi \equiv \frac{m-1}{2}N_j^L + mN_j^R + \frac{m-1}{2}N_{j+1}^L. \quad (6.56)$$

Using equations (6.17) and (6.18), we can compute the commutation relations between the fields to be

$$\left[ \partial_x \tilde{\theta}_l(x), \tilde{\varphi}_{l'}(x') \right] = im\pi\delta_{ll'}\delta(x-x'), \quad (6.57)$$

which again have the same structure as the original fields with a  $m$  additional factor.

Replacing the new field operators into (6.49), we have

$$\begin{aligned} \mathcal{O}_{l=j+1/2}^{1/m} &= \tilde{\psi}_{j+1,-}^\dagger \tilde{\psi}_{j,+} \\ &= \frac{1}{(2\pi\alpha)^2} e^{i(\tilde{\phi}_{j,+} - \tilde{\phi}_{j+1,-} + \pi\tilde{N}_j^\theta) + i\frac{\pi}{2}} = \frac{1}{(2\pi\alpha)^2} e^{2i(\tilde{\theta}_l + \frac{\pi}{4})}. \end{aligned} \quad (6.58)$$

Given a link  $l$ , the operator  $\mathcal{O}_{l=j+1/2}^{1/m}$  destroys a right moving fermion on the wire  $j$  while creates a left moving fermion on the wire  $j+1$ . We can observe the effect of the operator in the model defined in the links and the new variables in figure 6.3.

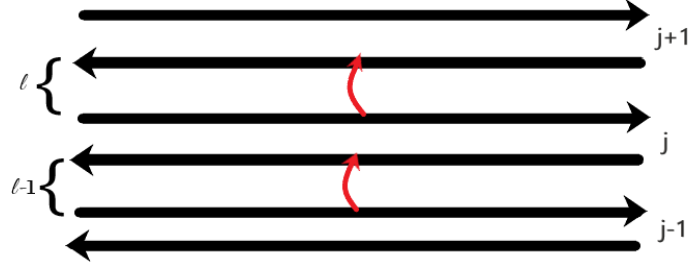


Figura 6.3: Schematic representation of the tunneling operator defined in the new variables and for the links  $l$ . Arrows pointing to the right represents right movers  $\tilde{\psi}_+$  and arrows pointing to the left represents left movers  $\tilde{\psi}_-$ . The effect of the operator is to destroy a right mover on a link  $l$  while creating a left mover on the same link.

Plugging equation (6.58) into (6.43), the Hamiltonian becomes

$$\begin{aligned} H &= H_0 + H_{fs} + H_{inter} \\ &= \tilde{H}_{SLL}[\tilde{\varphi}, \tilde{\theta}] + \sum_{l=1+1/2}^{\mathcal{N}-1/2} \int dx \frac{v^{1/m}}{\pi} \sin(2\tilde{\theta}_l). \end{aligned} \quad (6.59)$$

Analyzing the ground state of the Hamiltonian, there must be a constant configuration, let us say  $\tilde{\theta}_l = \tilde{\theta}^0$ , that minimizes the energy. The Hamiltonian is minimized when  $\tilde{\theta}^0 = -\frac{\pi}{2}$ , which implies a non-vanishing value of the field operator in the ground state. Therefore, we make a field redefinition as

$$\tilde{\theta}_l \rightarrow \tilde{\theta}_l + \frac{\pi}{2}, \quad (6.60)$$

so the Hamiltonian is given by

$$\begin{aligned} H &= H_0 + H_{fs} + H_{inter} \\ &= \tilde{H}_{SLL}[\tilde{\varphi}, \tilde{\theta}] - \sum_{l=1+1/2}^{\mathcal{N}-1/2} \int dx \frac{v^{1/m}}{\pi} \cos(2\tilde{\theta}_l). \end{aligned} \quad (6.61)$$

For this configuration, the energy is minimized when  $\tilde{\theta}_l = 0$ , which is a desired feature of the field operator in the ground state.

We shall analyze the Hamiltonian of the wires system in (6.61). When the coupling constant  $v^{1/m}$  is relevant, the system flows to a gapped phase at the low-energy limit. We can see this by expanding the cosine operator to quadratic terms, that is,

$$\cos(\tilde{\phi}_{j,+} - \tilde{\phi}_{j+1,-} + \pi \tilde{N}_j^\theta) = \frac{1}{2}(\tilde{\phi}_{j,+} - \tilde{\phi}_{j+1,-} + \pi \tilde{N}_j^\theta)^2 + \dots \quad (6.62)$$

The cosine operator couples right and left movers from neighbor wires, generating an energy gap in the bulk. The existence of the gapped phase is another indicative that this system captures

the Laughlin states physics.

So far we have seen that the coupled wires system with the tunneling operator in (6.46) leads to the fractional filling fraction and a gapped phase state, both features of the Hall system. To see that this configuration represents the Laughlin states, we need to check two other properties of the FQHE, namely, the bulk quasi-particles and the edge excitations.

### 6.5.2 Quasi-particles

The Hamiltonian in (6.61) is invariant under the transformation  $\tilde{\theta}_l \rightarrow \tilde{\theta}_l + \pi$ , as  $\tilde{H}_{SL\bar{L}}$  only has kinetic terms and the cosine operator remains constant when  $\tilde{\theta}_l$  jumps by  $\pi$ . This implies that the systems defined with  $\tilde{\theta}_l$  and  $\tilde{\theta}_l + \pi$  are equivalent. To analyze what this transformation might be related to, we consider to the charge density operator.

To compute the charge density operator related to the new variables, let us first check the algebra between the charge density operator in equation (6.30) and the original phase field, which is given by

$$[J_j(x), \varphi_{j'}(x')] = \left[ \frac{\partial_x \theta_j(x)}{\pi}, \varphi_{j'}(x') \right] = i\delta_{jj'}\delta(x - x'), \quad (6.63)$$

where we have used the commutation relation between the fields in (6.19). We can also integrate the current in the  $x$  coordinate to find the charge in each wire, so

$$\int dx [J_j(x), \varphi_{j'}(x')] = [Q_j, \varphi_{j'}(x')] = i\delta_{jj'}, \quad (6.64)$$

with  $Q_j$  being the charge operator for the wire  $j$ .

We want the Hamiltonian defined in the links to have the same charge conservation symmetry as the Hamiltonian defined in the wires, which is given by the algebra in (6.64). Thus, we need the charge density operator in the links  $\tilde{J}_l(x)$  to be such that

$$\int dx [\tilde{J}_l(x), \tilde{\varphi}_{l'}(x')] = [\tilde{Q}_l, \tilde{\varphi}_{l'}(x')] = i\delta_{ll'}. \quad (6.65)$$

Using the commutation relation in (6.57), we found the operator to be

$$\tilde{J}_l = \frac{\partial_x \tilde{\theta}_l}{m\pi}, \quad (6.66)$$

which satisfies equation (6.65).

From (6.66), it is straightforward that  $\partial_x \tilde{\theta}_l \rightarrow \partial_x \tilde{\theta}_l + \pi$  corresponds to a charge of  $\frac{1}{m}$ , the same charge as the bulk quasi-particles we found for the Hall fluid in section (5.6.3). When  $\tilde{\theta}_l$  jumps by  $\pi$ , it means that a quasi-particle is hopping from  $l = j - 1/2$  to  $l = j + 1/2$  [15]. The positive sign means that a unit of this fractional charge is being add to the link  $l = j + 1/2$ .

It is interesting that the quasi-particles are related to the links  $l$ , a region between two consecutive wires. These particles must be observed as topological excitations that permeate the sample. This argument, plus the fractional charge characteristic, is sufficient to ensure that the coupled wires system also exhibits the quasi-particle physics found in the Laughlin states. The last property to check is the existence of the edge states.

### 6.5.3 Edge States

Let us consider the coupled wires system in the new variables. Suppose that we have a finite number of wires, with  $j = 1$  and  $j = \mathcal{N}$  being the boundaries. We shall regard a simple forward interaction that leads the Hamiltonian in (6.33) to be

$$\tilde{H}_{SLL} = \frac{\lambda}{2\pi} \sum_{l=j+1/2} \int dx \frac{1}{g} (\partial_x \tilde{\theta}_l)^2 + g (\partial_x \tilde{\varphi}_l)^2, \quad (6.67)$$

where  $g$  is a constant and  $\lambda$  includes  $v_f$  and other constants. The full Hamiltonian then is given by

$$H = \tilde{\lambda} \int dx (\partial_x \tilde{\phi}_{1,-})^2 + (\partial_x \tilde{\phi}_{\mathcal{N},+})^2 + \frac{\lambda}{2\pi} \sum_{l=j+1/2} \int dx \frac{1}{g} (\partial_x \tilde{\theta}_l)^2 + g (\partial_x \tilde{\varphi}_l)^2 + \frac{v^{1/m}}{\pi} \cos(2\tilde{\theta}_l), \quad (6.68)$$

where  $\tilde{\lambda}$  is a coupling constant. The first two terms in the Hamiltonian represent the left mover on the first wire and the right mover on the last wire. Since we are not considering periodic boundary conditions, these modes have to be included into the Hamiltonian.

The chiral bosons  $\tilde{\phi}_{1,-}$  and  $\tilde{\phi}_{\mathcal{N},+}$  in (6.68) are not coupled with other movers by the cosine operator  $\cos(\tilde{\phi}_{j,+} - \tilde{\phi}_{j+1,-} + \pi \tilde{N}_j^\theta)$ . Therefore, these uncoupled operators are the only two gapless modes in the sample and represent the edge modes, while the rest of the bulk is gapped. We can write the Hamiltonian for the left mover as

$$H_{1,L} = \tilde{\lambda} \int dx (\partial_x \tilde{\phi}_{1,-})^2. \quad (6.69)$$

If we go back to the edge modes of the Laughlin states in equation (5.85), the Hamiltonian is

$$H_{edge} = \frac{vm}{4\pi} \int dx (\partial_x \phi)^2, \quad (6.70)$$

which has the same structure of (6.69). Also, we saw in equation (5.88) that the edge modes are represented by chiral bosons that only propagate in one way. These features allow us to affirm that the left mover of the first wire and the right mover on the last wire represent the gapless edge modes of the Laughlin states.

Besides the fractional filling fraction and the gapped bulk, the existence of quasi-particles and edge modes in the coupled quantum wires system proves that this approach captures the physics of the fractional quantum Hall effect. We started with electronic degrees of freedom and end up describing features in the same way as described by the effective Chern-Simons theory, which indicates a direct connection between the wires and the effective approach.

We have seen that the edge states from both theories already had the same structure. It was not quite clear how a chiral boson described the edge theory of a fermionic problem in the effective description, but in the wires approach, this feature arose naturally since we are using the bosonization technique. Besides representing the same physics, we want to identify a map between the quantum wires approach and the effective Chern-Simons theory. The next chapter is dedicated to proving that when we implement the continuum limit in the  $y$  direction for the wires system, we are able to find a one to one correspondence between the wire operators and the operators from the Chern-Simons theory. This correspondence will lead to a connection between microscopic and effective degrees of freedom.

## 7 FROM QUANTUM WIRES TO CHERN-SIMONS THEORY

We have seen that the quantum wires description captures the fractional quantum Hall effect physics. Our goal is to find a map between the operators in the quantum wires approach and the gauge field operators in the effective description of the Laughlin states. Being the quantum wires essentially a microscopic approach, the map between the operators will provide a direct bridge between the microscopic and the effective degrees of freedom.

We follow references [18] and [19], which have successfully studied this connection. We start with the results from chapter 6 and choose a specific type of forward scattering. Then we shall consider a continuum low-energy limit in the wires system so that it is properly defined in 2+1 dimensions, allowing us to compare this system with the Chern-Simons theory in 2+1 dimensions. The final step is to build an equivalence between the wire operators and the gauge field operators and analyze the effective action, to compare the results with the Chern-Simons theory in chapter 5.

### 7.1 THE FORWARD SCATTERING INTERACTION

The Hamiltonian of the coupled wires system that leads to the Laughlin states is

$$\begin{aligned} H &= H_0 + H_{fs} + H_{inter} = H_{SLL} + \sum_j \int dx (v^{m_a n_a} \mathcal{O}_j^{1/m}(x) + h.c.) \\ &= H_{SLL} - \sum_j \int dx \frac{v^{1/m}}{\pi} \sin(\varphi_j - \varphi_{j+1} + m(\theta_j + \theta_{j+1}) - \pi(N_j^R + N_j^L)), \end{aligned} \quad (7.1)$$

where we have used the explicit form of the tunneling operator given by (6.49) and introduced a minus sign that can be incorporated in  $v^{1/m}$ . In the last chapter, we only assumed that the forward scattering interactions were chosen in such way to make the tunneling operator relevant. Here we choose a precise form of the operator.

We take the forward scattering to be restricted between densities on the same wire, that is, for  $V^{jk} = \delta^{jk}$  in equation (6.32). Therefore, the most general interaction that respects charge conservation in each wire is given by

$$\begin{aligned} H_{fs} &= \sum_{jk, pp'} \int dx \psi_{j,p}^\dagger \psi_{j,p} V_{pp'}^{jk} \psi_{k,p'}^\dagger \psi_{k,p'} \\ &= \sum_j \int dx \frac{\lambda_a}{2} [(J_j^R)^2 + (J_j^L)^2] + \lambda_b J_j^R J_j^L \\ &= \sum_j \frac{1}{4\pi} \int dx (\lambda_a - \lambda_b) (\partial_x \varphi_j)^2 + (\lambda_a + \lambda_b) (\partial_x \theta_j)^2, \end{aligned} \quad (7.2)$$

where  $\lambda_a$  and  $\lambda_b$  are coupling constants and  $J_j^{R/L}$  are the currents defined in (6.30). This precise interaction will lead to terms in the action that will make the connection with the Chern-Simons theory in a straightforward manner. In the end of this chapter, we discuss if this is the only structure allowed or if the connection is valid for general forward interactions. Forward interactions are expected to affect only the dynamics of the system, not the ground state properties, which indicates that the form of this interaction should not restrict the connection, as we will discuss later.

Using equations (6.29), (7.1) and (7.2), we may write the Hamiltonian as

$$\begin{aligned} H &= H_0 + H_{fs} + H_{inter} \\ &= \sum_j \int dx \frac{1}{2\pi} \left[ (v_F + \frac{\lambda_a}{2\pi} - \frac{\lambda_b}{2\pi})(\partial_x \varphi_j)^2 + (v_F + \frac{\lambda_a}{2\pi} + \frac{\lambda_b}{2\pi})(\partial_x \theta_j)^2 \right] \\ &\quad - \frac{v^{1/m}}{\pi} \sin(\varphi_j - \varphi_{j+1} + m(\theta_j + \theta_{j+1}) - \pi(N_j^R + N_j^L)). \end{aligned} \quad (7.3)$$

To simplify the Hamiltonian, we perform a change of variables where

$$\tilde{v} \equiv \sqrt{(v_F + \frac{\lambda_a}{2\pi})^2 - (\frac{\lambda_b}{2\pi})^2}, \quad (7.4)$$

and

$$K \equiv \frac{(v_F + \frac{\lambda_a}{2\pi} - \frac{\lambda_b}{2\pi})}{(v_F + \frac{\lambda_a}{2\pi} + \frac{\lambda_b}{2\pi})}, \quad (7.5)$$

so that the equation (7.3) becomes

$$\begin{aligned} H &= \sum_j \int dx \frac{\tilde{v}}{2\pi} \left[ K(\partial_x \varphi_j)^2 + \frac{1}{K}(\partial_x \theta_j)^2 \right] \\ &\quad - \frac{v^{1/m}}{\pi} \sin(\varphi_j - \varphi_{j+1} + m(\theta_j + \theta_{j+1}) - \pi(N_j^R + N_j^L)). \end{aligned} \quad (7.6)$$

Also, we implement a rescale of the field variables in order to bring the kinetic terms in the canonical form,

$$\varphi_j \rightarrow \frac{1}{\sqrt{K}}\varphi_j, \quad \theta_j \rightarrow \sqrt{K}\theta_j, \quad (7.7)$$

which do not change their commutation rules,

$$[\theta_j(x), \varphi_{j'}(x')] \rightarrow \left[ \sqrt{K}\theta_j(x), \frac{1}{\sqrt{K}}\varphi_{j'}(x') \right] = i\pi\delta_{jj'}\Theta(x - x'). \quad (7.8)$$

Implementing the rescale into (7.6), the Hamiltonian is given by

$$H = \sum_j \int dx \frac{\tilde{v}}{2\pi} [(\partial_x \varphi_j)^2 + (\partial_x \theta_j)^2] - \frac{v^{1/m}}{\pi} \cos \left( \frac{1}{\sqrt{K}} (\varphi_j - \varphi_{j+1}) + m\sqrt{K} (\theta_j + \theta_{j+1}) - \pi\sqrt{K} (N_j^R + N_j^L) \right). \quad (7.9)$$

In the above, following the same argument that led to equation (6.60), we made the variable change  $\theta_j \rightarrow \theta_j - \frac{\pi}{4m\sqrt{K}}$ , so the Hamiltonian is minimized for  $\theta_j = 0$ .

We intend to take the continuum limit in the wires system so that the model is properly defined in 2+1 dimensions, as the Chern-Simons theory. Since this limit is related to low-energy physics, it is important that we analyze the Hamiltonian under the renormalization group. In section 6.5, we supposed that the interwire operator was relevant at low-energies due to a general form of the forward interaction. With the specific form in (7.2), we must analyze whether the cosine operator is relevant or irrelevant in the renormalization group sense.

## 7.2 RENORMALIZATION GROUP

As our Hamiltonian has the same structure as the Sine-Gordon model, we follow the same path realized in section 4.4.1, so we focus only in some general steps. The action of the coupled wires system in 1+1 dimensions is given by

$$S = \sum_j \int d^2x \frac{\tilde{v}}{2\pi} ((\partial_x \theta_j)^2 - (\partial_x \varphi_j)^2) + \frac{v^{1/m}}{\pi} \cos \left( \frac{1}{\sqrt{K}} (\varphi_j - \varphi_{j+1}) + m\sqrt{K} (\theta_j + \theta_{j+1}) - \pi\sqrt{K} (N_j^R + N_j^L) \right). \quad (7.10)$$

As  $\partial_x \theta_j = \Pi = \dot{\varphi}$ , we only need a momentum cut-off for the phase field, which is implemented by

$$\varphi_j = \varphi_{j,s} + \varphi_{j,f} = \varphi_j(0 \leq k \leq \Lambda(1 - d\mu)) + \varphi_j(\Lambda(1 - d\mu) < k \leq \Lambda), \quad (7.11)$$

being  $f$  the fast modes,  $s$  the slow modes and  $\mu$  a rescaling factor related to the energy.

After implementing the momentum cut-off, we can integrate out fast modes from the partition function of the system, which leads to

$$Z = \int d\varphi_{j,s} \exp \left[ i \sum_j \int \left[ d^2x \frac{\tilde{v}}{2\pi} ((\partial_x \theta_{j,s})^2 - (\partial_x \varphi_{j,s})^2) + \frac{v^{1/m}}{\pi} \cos(A_s) \langle \cos(A_f) \rangle_f \right] \right], \quad (7.12)$$

where we used the short notation  $A \equiv \frac{1}{\sqrt{K}} (\varphi_j - \varphi_{j+1}) + m\sqrt{K} (\theta_j + \theta_{j+1}) - \pi\sqrt{K} (N_j^R + N_j^L)$ .

The expected value of the cosine operator is

$$\begin{aligned}
\langle \cos(A_f) \rangle_f &= e^{-\frac{1}{2}\langle AA \rangle_f} \\
&= \exp \left[ \left( \frac{1}{K} + m^2 K \right) \int_{\Lambda(1-d\mu)}^{\Lambda} \int_0^{2\pi} \frac{k dk d\theta}{4\pi} \frac{1}{k^2} \right] \\
&= 1 - \frac{1}{2} \left( \frac{1}{K} + m^2 K \right) d\mu.
\end{aligned} \tag{7.13}$$

In the above, we used the fact that  $\varphi$  and  $\theta$  have equal correlation functions. To use the same results as in (4.49), we had to multiply the correlations by  $\frac{4\pi}{2}$  because of the definitions of the fields in (6.14), (6.15) and (6.16).

We now perform a coordinate rescale

$$d^2 x = \frac{1}{s^2} d^2 x' = (1 + 2d\mu) d^2 x', \tag{7.14}$$

where  $s = 1 - d\mu$ . With the integration of fast modes and the coordinate rescale, the partition function becomes

$$Z = \int d\varphi_j \exp \left[ i \int \left[ d^2 x \frac{\tilde{v}}{2\pi} ((\partial_x \theta_j)^2 - (\partial_x \varphi_j)^2) + \frac{v^{1/m}}{\pi} \left( 1 + \left( 2 - \frac{1}{2} \left( \frac{1}{K} + m^2 K \right) \right) d\mu \right) \cos(A) \right] \right]. \tag{7.15}$$

Comparing the cosine operator in (7.10) with (7.15), we find the beta function to be

$$\frac{dv^{1/m}}{d\mu} = \left( 2 - \frac{1}{2} \left( \frac{1}{K} + m^2 K \right) \right) v^{1/m}. \tag{7.16}$$

Considering positive values of  $K$  and odd integer values for  $m$ ,  $\frac{1}{2} \left( \frac{1}{K} + m^2 K \right)$  can never be greater than 2, which implies that the cosine operator is irrelevant at low-energies. Hence, this operator would not generate an energy gap in the system at a low-energy limit, which is inconsistent with the analysis of the quantum wires system representing the gapped Laughlin states. To avoid this issue, we are going to define the system to be in a strong coupling regime. In this regime, we take the system to be strongly coupled at large distances, which implies that we must consider the interaction term. In this case, there is no need for the RG to evaluate the relevance of the interaction.

Instead of demanding a relevant operator after the renormalization group, we are going to consider  $v^{1/m} \rightarrow \infty$ . This implies that the operator actually generates the gapped phase at low energies since the system is strongly correlated and we shall not disregard the cosine operator. This requirement leads the system to be gapped at the low-energy limit and solves our problem, but it also implies that we cannot treat the tunneling interaction perturbatively, so we must be careful when evaluating the energy scales of the system. We may now implement the continuum limit and analyze the wire operators algebra, in order to make the connection

with the effective theory.

### 7.3 THE CONTINUUM LIMIT

We need to define the quantum wires in the continuum limit in the  $y$  direction, to relate its algebra with the effective theory. We first investigate the wires operator algebra in the discrete and afterwards we take the continuum limit. Let us define the following operators

$$\mathcal{O}_1^j \equiv \sqrt{\frac{1}{\pi}} \partial_x \theta_j, \quad (7.17)$$

$$\mathcal{O}_2^j \equiv \sqrt{\frac{1}{\pi}} \partial_x \varphi_j, \quad (7.18)$$

and

$$\mathcal{O}_3^j \equiv \frac{1}{\sqrt{K}} (\varphi_j - \varphi_{j+1}) + m\sqrt{K}(\theta_j + \theta_{j+1}) - \pi\sqrt{K}(N_j^R + N_j^L). \quad (7.19)$$

Their commutation relations are given by

$$\begin{aligned} [\mathcal{O}_1^j(x), \mathcal{O}_2^{j'}(x')] &= \frac{1}{\pi} \partial_{x'} [\partial_x \theta_j(x), \varphi_{j'}(x')] \\ &= i\delta_{jj'} \partial_{x'} \delta(x - x') = -i\delta_{jj'} \partial_x \delta(x - x'), \end{aligned} \quad (7.20)$$

$$\begin{aligned} [\mathcal{O}_1^j(x), \mathcal{O}_3^{j'}(x')] &= \sqrt{\frac{1}{K\pi}} [\partial_x \theta_j(x), \theta_{j'}(x') - \varphi_{j'+1}(x')] - \sqrt{K\pi} [\partial_x \theta_j(x), N_{j'}^R + N_{j'}^L] \\ &= i\sqrt{\frac{\pi}{K}} (\delta_{jj'} - \delta_{j,j'+1}) \delta(x - x'), \end{aligned} \quad (7.21)$$

and

$$\begin{aligned} [\mathcal{O}_2^j(x), \mathcal{O}_3^{j'}(x')] &= -m\sqrt{\frac{K}{\pi}} [\partial_x \varphi_j(x), \theta_{j'}(x') + \theta_{j'+1}(x')] - \sqrt{K\pi} [\partial_x \varphi_j(x), N_{j'}^R + N_{j'}^L] \\ &= -im\sqrt{K\pi} (\delta_{jj'} + \delta_{j,j'+1}) \delta(x - x'), \end{aligned} \quad (7.22)$$

with other commutation relations vanishing.

To implement the continuum limit in the  $y$  direction, we must study the system in large distance physics regime (low energies), so the separation  $a$  between the wires is small enough to make the system continuum. This is explicitly given by

$$\theta_j(t, x) \rightarrow \sqrt{a}\theta(t, x, y), \quad \varphi_j(t, x) \rightarrow \sqrt{a}\varphi(t, x, y), \quad (7.23)$$

and

$$\delta_{jj'} = a\delta(y - y'), \quad (7.24)$$

where  $a$  is the separation between consecutive wires. Also,  $\delta(r - r') = \delta(x - x')\delta(y - y')$ . Since  $a$  is our length scale, the energy scale is  $\Lambda = \frac{1}{a}$  and we are considering energies  $E \ll \Lambda$ .

Taking into account the rescaling above, the commutation rules become

$$[\mathcal{O}_1(r), \mathcal{O}_2(r')] = -i\partial_x\delta(x - x')\delta(y - y') = -i\partial_x\delta(r - r'), \quad (7.25)$$

$$\begin{aligned} [\mathcal{O}_1(r), \mathcal{O}_3(r')] &= i\sqrt{\frac{\pi}{K}}(\delta(y - y') - \delta(y - y' + a))\delta(x - x') \\ &= -ia\sqrt{\frac{\pi}{K}}\partial_y\delta(y - y')\delta(x - x') = -ia\sqrt{\frac{\pi}{K}}\partial_y\delta(r - r'), \end{aligned} \quad (7.26)$$

and

$$\begin{aligned} [\mathcal{O}_2(r), \mathcal{O}_3(r')] &= -im\sqrt{K\pi}(\delta(y - y') + \delta(y - y' + a))\delta(x - x') \\ &= -im\sqrt{K\pi}(\delta(y - y') + \delta(y - y' + a) + \delta(y - y') - \delta(y - y'))\delta(x - x') \\ &= -2im\sqrt{K\pi}\delta(y - y')\delta(x - x') - 2aim\sqrt{K\pi}\partial_y\delta(y - y')\delta(x - x') \\ &= -2im\sqrt{K\pi}\delta(r - r') - 2aim\sqrt{K\pi}\partial_y\delta(r - r'). \end{aligned} \quad (7.27)$$

Equations (7.25), (7.26) and (7.27) represent the algebra of the coupled wires system in the continuum limit. The Maxwell-Chern-Simons theory has a similar algebra, as we shall now study.

#### 7.4 THE MAXWELL-CHERN-SIMONS ALGEBRA

We want to investigate the Maxwell-Chern-Simons (MCS) algebra [43], which is the Chern-Simons theory with the presence of Maxwell's term. In chapter 5 we defined the Chern-Simons theory in the absence of the Maxwell term  $F^{\mu\nu}F_{\mu\nu}$  since it was not relevant at low-energy physics. In this section, we must consider this term in order to connect the MCS algebra with the quantum wires algebra. At the end, we are going to analyze whether the Maxwell term can be neglected or not.

In 2+1 dimensions, the MCS Lagrangian is

$$\mathcal{L} = \frac{\mu}{2}\epsilon^{\mu\nu\rho}\partial_\mu A_\nu A_\rho - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (7.28)$$

where  $\mu$  is related to the mass gap. The Hamiltonian is given by

$$H = \int dx dy \frac{1}{2} (\vec{E}^2 + B^2), \quad (7.29)$$

with  $\vec{E}^2 = E_x^2 + E_y^2$ . Following reference [43], the commutation relations in the Maxwell-Chern-Simons theory are

$$[\Pi_i(r), A_j(r')] = -i\delta_{ij}\delta(r - r'), \quad (7.30)$$

with  $\Pi^i$  being the conjugated momentum given by

$$\Pi^i = \frac{\delta L_{MCS}}{\delta \dot{A}_i} = -E_i + \frac{\mu}{2}\epsilon_{ij}A_j, \quad E_i = \dot{A}_i. \quad (7.31)$$

In the above,  $i, j = x, y$ . Having the commutation relation for the theory, we may analyze the connection with the wires system.

To observe the similarity between the MCS algebra and the commutation relations for the wire operators, we shall consider the commutation rules for the electric and magnetic fields in the MCS theory, that is,

$$\begin{aligned} [E_i(r), E_j(r')] &= \left[ \frac{\mu}{2}\epsilon_{ij}A_j(r) - \Pi_i(r), \frac{\mu}{2}\epsilon_{jk}A_k(r') - \Pi_j(r') \right] \\ &= -\left[ \frac{\mu}{2}\epsilon_{ij}A_j(r), \Pi_j(r') \right] + \left[ \frac{\mu}{2}\epsilon_{jk}A_k(r'), \Pi_i(r) \right] \\ &= -i\frac{\mu}{2}\epsilon_{ij}\delta(r - r') + i\frac{\mu}{2}\epsilon_{jk}\delta(r - r') \\ &= -i\mu\epsilon_{ij}\delta(r - r'), \end{aligned} \quad (7.32)$$

and

$$\begin{aligned} [B(r), E_j(r')] &= \left[ \epsilon_{ij}\partial_i A_j(r), \frac{\mu}{2}\epsilon_{jk}A_k(r') - \Pi_j(r') \right] \\ &= -\epsilon_{ij}\partial_i [A_j(r), \Pi^j(r')] = -i\epsilon_{ij}\partial_i\delta(r - r'), \end{aligned} \quad (7.33)$$

where we have used  $B = \epsilon_{ij}\partial_i A_j$  for the gauge  $A_0 = 0$ . It is evident that the relation in (7.33) has the same structure as the relations in (7.25) and (7.26). These results indicate that we must connect the wire operators to the magnetic and electric fields.

## 7.5 CONNECTION BETWEEN THE ALGEBRAS

We identify the wire operators as

$$\mathcal{O}_1(t, r) \equiv B(t, r), \quad \mathcal{O}_2(t, r) \equiv E_y(t, r), \quad \mathcal{O}_3(t, r) \equiv -a\sqrt{\frac{\pi}{K}}E_x(t, r). \quad (7.34)$$

This is the main step to connect the wires approach to the effective theory. It is important to notice that the equalities in (7.34) are identifications between physical quantities [19]. The wire operators  $\mathcal{O}_{1,2,3}$  are electron operators, so it is natural that we make the equivalence between these operators and the magnetic and electric fields, which are also physical quantities. Using the new identifications above, the equal-time algebra of the operators  $\mathcal{O}_{1,2,3}$  is written as

$$[\mathcal{O}_1, \mathcal{O}_{(2,3)}] \rightarrow [B(t, r), E_j(r')] = -i\epsilon_{jk}\partial_k\delta(r - r'), \quad (7.35)$$

$$[\mathcal{O}_2, \mathcal{O}_3] \rightarrow [E_y(r), E_x(r')] = -2im\frac{K}{a}\delta(r - r') - 2imK\partial_y\delta(r - r'), \quad (7.36)$$

where  $j, k = x, y$  and  $\epsilon_{xy} = 1$ . The relations in equation (7.35) are identical to equation (7.33). For the wires system to possess the MCS algebra, we need that the relation in (7.36) to be equal to (7.32).

We must be careful with the commutation rule in (7.36) where there is the factor  $\frac{K}{a}$ . Since  $a \rightarrow 0$ , we must define the coupling constants  $\lambda_a$  and  $\lambda_b$  so  $K$ , given by (7.5), is also small. This requirement leads  $\frac{K}{a}$  to be a finite quantity smaller than our energy scale, that is,  $\frac{K}{a} \ll \Lambda$ . As the parameters  $\lambda_a$  and  $\lambda_b$  are only related to interactions inside the wires,  $\frac{K}{a}$  may be related to a gap for excitations propagating only parallel to the wires in the  $x$  direction [18]. We then define a new parameter

$$\Delta_1 \equiv \frac{K}{a}. \quad (7.37)$$

Taking  $K$  to be small and  $\Delta_1$  to be a finite quantity, the commutation relation in (7.36) becomes

$$[E_x(r), E_y(r')] \approx -2im\Delta_1\delta(r - r'). \quad (7.38)$$

This relation is equal to the commutator in (7.32) with

$$\mu = 2m\Delta_1. \quad (7.39)$$

We can now confirm that the coupled wires system respect the MCS algebra with  $\mu$  defined in equation (7.39). The last step needed to connect the coupled wires system to the Chern-Simons theory is the replacement of the continuum wire operators into the Hamiltonian, so that we can write the action of the system and compare to the Chern-Simons action.

## 7.6 THE MAXWELL-CHERN-SIMONS HAMILTONIAN

Replacing the operators from (7.34) into (7.9), with the appropriate factors displayed in (7.23), the Hamiltonian becomes

$$\begin{aligned}
H &= \sum_j \int dx \frac{a\tilde{v}}{2}(E_y^2 + B^2) - \frac{v^{1/m}}{\pi} \cos\left(a^{3/2} \sqrt{\frac{\pi}{K}} E_x\right) \\
&= \int dx dy \frac{\tilde{v}}{2}(E_y^2 + B^2) - \frac{v^{1/m}}{\pi a} \cos\left(a^{3/2} \sqrt{\frac{\pi}{K}} E_x\right) \\
&= \int dx dy \frac{\tilde{v}}{2}(E_y^2 + B^2) + \frac{v^{1/m}}{\pi a} \cos\left(-a^{3/2} \sqrt{\frac{\pi}{K}} E_x\right) \\
&= \int dx dy \frac{\tilde{v}}{2}(E_y^2 + B^2) + \frac{1}{2} \frac{av^{1/m}}{\Delta_1} E_x^2 - \dots
\end{aligned} \tag{7.40}$$

In the above we made the replacement  $\sum_j a \rightarrow \int dy$  and used equation (7.37). In addition, we expanded the cosine function in a Taylor series and used the fact that the cosine is an even function, that is,  $\cos(a) = \cos(-a)$ .

In (7.40) there is a new energy scale characterized by  $\frac{av^{1/m}}{\Delta_1}$ , such that we define a new parameter

$$av^{1/m} \equiv \Delta_2. \tag{7.41}$$

This parameter is in agreement with our initial assumption that the system is strongly coupled at low energies ( $v^{1/m} \rightarrow \infty$ ), as we can demand  $\Delta_2$  to assume finite values when  $a \rightarrow 0$ . The parameter  $\Delta_2$  involves only the coupling constant of the tunneling operator, which means that this energy scale is related to gap for excitations moving perpendicular to the wires [18].

For the Hamiltonian in (7.40) to represent the MCS Hamiltonian, we need

$$\frac{\Delta_2}{\Delta_1} = \tilde{v}, \tag{7.42}$$

so the Hamiltonian becomes

$$H = \int dx dy \frac{\tilde{v}}{2}(\vec{E}^2 + B^2), \tag{7.43}$$

where  $\vec{E}^2 = E_x^2 + E_y^2$  and we have only considered the first term of the Taylor expansion in (7.40), since the continuum limit implies that we are at low-energy physics where high-order terms may be discarded. This Hamiltonian is the MCS Hamiltonian displayed in (7.29) with a multiplying constant.

Before computing the action of the system, we need to analyze whether the energy scales in the system are well defined. We considered  $\Delta_1$  to be a finite quantity, which implied in  $K \rightarrow 0$ . Then, we also required  $\Delta_2$  to be finite since  $v^{1/m} \rightarrow \infty$ . Now we must

evaluate the equality in (7.42) and compare if all these parameters are simultaneously well defined. The parameter  $\tilde{v}$  depends on  $\Delta_1$  and  $\Delta_2$ , which are respectively related to the energy gaps in the  $x$  direction and in the  $y$  direction. We expect the system to have an isotropic motion of charges in 2+1 dimensions, where there should not be a direction in which the particles move preferentially. For this to be true, we need that  $\Delta_1 \sim \Delta_2$ , which implies  $\tilde{v} \sim 1$ . Thus, we must evaluate if  $\tilde{v} \sim 1$  can be simultaneously met with  $K \rightarrow 0$ , so all the parameters are well defined.

If we choose  $\tilde{v} \equiv 1$ , equations (7.4) and (7.5) become

$$(v_F + \frac{\lambda_a}{2\pi}) = \pm \sqrt{1 + \frac{\lambda_b}{(2\pi)^2}}, \quad (7.44)$$

and

$$K = \sqrt{\frac{\sqrt{1 + \frac{\lambda_b}{(2\pi)^2}} - \frac{\lambda_b}{2\pi}}{\sqrt{1 + \frac{\lambda_b}{(2\pi)^2}} + \frac{\lambda_b}{2\pi}}}. \quad (7.45)$$

We can observe that  $K$  decreases with an increase of  $\lambda_b$ . We can let  $\lambda_b$  to be large enough so that  $K$  is small enough, so  $\tilde{v} \sim 1$  and  $K \rightarrow 0$  are simultaneously met. Thus, the energy-scales of the system are well defined and we may consider the Hamiltonian for  $\tilde{v} \equiv 1$  and  $K \rightarrow 0$ , given by

$$H = \int dx dy \frac{1}{2} (\vec{E}^2 + B^2). \quad (7.46)$$

This is exactly the Hamiltonian of the MCS theory. We shall compute the action of the system.

Since the operators in (7.46) respect the MCS algebra with  $\mu = 2m\Delta_1$ , we use equation (7.28) and write the Lagrangian and action for the coupled wires system as

$$\mathcal{L} = \int dx dy m\Delta \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \frac{F^{\mu\nu} F_{\mu\nu}}{4}, \quad (7.47)$$

and

$$S = \int d^3x m\Delta \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \frac{F^{\mu\nu} F_{\mu\nu}}{4}. \quad (7.48)$$

As  $\tilde{v} = 1$ , we are using  $\Delta_1 = \Delta_2 = \Delta$ . To compare this action to the Chern-Simons action in (5.65), we need to perform a last field rescaling so we can eliminate the parameter  $\Delta$  from the gauge field and from the Dirac quantization of the system.

To find the desired factor of the rescaling, we start with the charge of the

system. We defined the total charge density operator of the wires system as

$$J = \sum_j J_j = \frac{\partial_x \theta_j(x)}{\pi}. \quad (7.49)$$

After performing the field rescaling in (7.7) and applying the continuum limit in (7.23), the operator becomes

$$J = \int dy \sqrt{\frac{K}{a}} \frac{\partial_x \theta(r)}{\pi} = \int dy \sqrt{\Delta} \frac{\partial_x \theta(r)}{\pi}. \quad (7.50)$$

Now we can compute the total charge of the system as

$$\begin{aligned} Q &= \int dx J = \int dx dy \sqrt{\Delta} \frac{\partial_x \theta(r)}{\pi} \\ &= \int d^2x \sqrt{\frac{\Delta}{\pi}} B(t, r). \end{aligned} \quad (7.51)$$

With the rescaling

$$B \rightarrow \frac{1}{\sqrt{4\pi\Delta}} B, \quad (7.52)$$

and

$$A \rightarrow \frac{1}{\sqrt{4\pi\Delta}} A, \quad (7.53)$$

we obtain

$$Q = \frac{1}{2\pi} \int d^2x B(t, r) \propto \mathbb{Z}. \quad (7.54)$$

The total charge of the system is proportional to an integer number, since it must be a multiple of the electron charge. This is the same relation we had in equation (5.56) where the minimal flux of the system could only be proportional to one electron charge, respecting the Dirac's quantization rule. It is important that both the effective C.S. theory and the quantum wires theory have the same flux quantization so that we can evaluate if both theories capture equivalent physics. Plugging (7.53) into the action in (7.48),

$$S = \int d^3x \frac{m}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \frac{1}{16\Delta} F^{\mu\nu} F_{\mu\nu}, \quad (7.55)$$

with

$$\frac{1}{2\pi} \int d^2x B(t, r) \in \mathbb{Z}. \quad (7.56)$$

Considering the limit of low energies ( $E \ll \Lambda$ ), we can neglect the Maxwell term in (7.55) since the leading contribution for the effective theory is the Chern-Simons term. In this limit, our action is identical to the Chern-Simons action in (5.65), that is,

$$S = \int d^3x \frac{m}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \quad (7.57)$$

Finally, we have shown that by choosing specific types of interactions, the quantum wires approach can be directly related to the effective approach described in chapter 5. We have already seen that the edge modes in the coupled wires system had the same structure as the edge modes in the Chern-Simons theory. In this chapter, we have also made a connection between the bulk actions of both approaches. There are some points that are worth mention about this connection.

Going back to chapter 6, we described the quantum wires approach to be a microscopic approach, since we started with electronic degrees of freedom. After introducing specific interactions to couple the system and considering the continuum limit, we were able to show that this theory can also capture the topological features of the Chern-Simons theory, represented by the action in (7.57).

When we neglect the Maxwell term, the action is governed by the topological Chern-Simons term, where the only microscopic parameter left is the integer number  $m$ . One may think that this parameter could also be eliminated through a field rescale such as in (7.53), but this is not possible as a rescaling would leave either the flux relation in (7.56) or the action in (7.57) with a  $m$  factor [18].

The connection between the quantum wires and the effective approach has proven to be the missing link between the effective and the microscopic degrees of freedom. It was quite unclear how these degrees of freedom were connected. Now, we can directly trace back the microscopic parameter  $m$  to the interwire interaction of the wires system. It is possible to propose other Hamiltonians with distinct microscopic interactions and analyze its effects in the effective theory, which was not possible in the other approaches.

We should ask ourselves if our results are merely coincidences of our forward interaction choice or if this approach is actually valid for general interactions. Had we chosen a different forward interaction in (7.2), its effect would be to change the Maxwell term in (7.55), since the microscopic parameters such as the energy gaps and the coupling constant are encoded in  $\Delta_1 = \Delta_2 = \Delta$ . These parameters only affect the Maxwell term, related to the dynamics of collective excitations [18]. Different types of interaction would only change this term, but not the dominant topological character of our action, described by the Chern-Simons term. At the low-energy limit, this dynamical term does not play any role in our theory, which implies that our approach is valid. However, there is no guarantee that other forward interactions may lead to direct equivalences between the bosonic wire operators and the gauge field operators.

Lastly, when we described the edge states in chapter 5, we observed that there were chiral boson fields at the edge of the system, which was a hint of the possibility of in-

roducing the bosonization technique to study the quantum Hall system. After applying this technique to the quantum wires, we were able to connect the boson fields directly with the effective Chern-Simons theory in (7.34), which once again has shown the usefulness of the bosonization technique.

The connection studied in this chapter was strictly done for the quantum Hall states, which is characterized by an Abelian gapped phase. As shown in references [15, 16, 17], the quantum wires approach describes several types of Abelian and non-Abelian gapped phases. The connection between the quantum wires description of these different gapped phases and the effective theory may still be evaluated.

## 8 FINAL REMARKS

The main goal of this work was to apply the bosonization technique to the quantum Hall system. To acquire familiarity with the problem, we first studied the quantum Hall system with two usual approaches: *i*) the wave function approach, and *ii*) the effective field theory approach. Since the microscopic theory was not approached by any analytical method, it was not possible to find an exact solution for the problem, being the Laughlin wave function an approximate solution. We then studied the effective Chern-Simons theory, which is a topological theory that successfully explains the Hall system from a macroscopic point of view. However, this effective theory did not connect the effective action with the microscopical degrees of freedom, which led us to our final approach, the quantum wires.

In the quantum wires approach, we described the Hall system as an array of fermionic systems in (1+1) dimensions with interactions between them. After applying the bosonization formulas, we have seen that this description captures the physical properties of the quantum Hall system, and therefore it is a valuable description for the effect. Also, with the bosonization technique, we were able to study the system in the strongly coupled regime.

An interesting point is that the quantum wires approach describes the edge modes and the quasi-particle excitations in a very similar way described by the effective CS theory. In chapter 6, we have also observed that the edge Hamiltonian for the wires system had the same structure as the edge Hamiltonian in the effective approach. The fact that both approaches similarly described the quantum Hall system physics motivated us to pursue a connection between the quantum wires and the effective Chern-Simons theory.

In chapter 7, after choosing specific forward interactions and taking the strongly coupled regime, we identified a direct map between the coupled wires and the Chern-Simons theory. We observed that the wire bosonic operators respected the same algebra as the Maxwell-Chern-Simons theory, which resulted in an equivalence between the wires operators and the electric and magnetic fields (gauge-invariant objects). At the low-energy limit, this connection between the operators allowed us to identify the quantum wires action as the Chern-Simons action. The connection between both theories demonstrated that the microscopic degrees of freedom are directly related to the effective description by the integer parameter  $m$  left in the effective action. This correspondence between the degrees of freedom was not well defined in the microscopic and effective approaches, which is an advantage of the method studied in this thesis.

To conclude, it is worth mentioning that the Ref. [44] presents a different way to connect the coupled wires system and the effective field theory. In [44], after a proper choice of gauge, the bosonic wire operators were identified with a 2-dimensional gauge field. It was observed that these fields satisfy the same commutation relations as the gauge field in the Chern-Simons theory. Also, by relaxing the condition of the strong coupling limit, this

paper showed how the coupled wires Lagrangian became the Chern-Simons-Landau-Ginsburg (CSLG) Lagrangian. As the effective CSLG theory involves composite boson terms in the Lagrangian, a starting point to connect the approach in [44] with the approach in this thesis may be to use a particle-vortex duality, where a fermion can be described as a boson with flux attachment [45]. The equivalence between these different approaches is yet to be studied.

## A APPENDIX A: PATH INTEGRAL VERSUS PARTITION FUNCTION

In QFT we define the theory in the path integral approach, using time as the parameter, while the partition function in statistical mechanics is defined using temperature. We want to make a connection between these two definitions by considering the partition function at a finite temperature. A more complete derivation of this topic can be found in references [1] and [5]. The statistical mechanics partition function is

$$Z[\beta] = \text{tr}(e^{-\beta H}), \quad (\text{A.1})$$

being  $\beta = \frac{1}{T}$ ,  $T$  the temperature and  $H$  the Hamiltonian.

In quantum mechanics and QFT, we use time as the parameter in the path integral, so the Feynman path integral is

$$\langle q_i | e^{-iHt} | q_f \rangle = \int_{q(0)=q_i}^{q(t)=q_f} Dq e^{iS} \quad (\text{A.2})$$

where  $t$  is the time,  $q_i$  is the position of the particle in  $t = 0$  and  $q_f$  is the position at a later time  $t$ . In order to make equations (A.1) and (A.2) equivalent, we need to perform three steps. First we execute a wick rotation, which means going to the euclidean time where  $it = \tau$ . Second we make the requirement that  $\beta = \tau$ . Last, we impose that  $q(0) = q(\beta)$  in order to get the trace present in equation (A.1). This last equality implies periodic conditions for the euclidean time, that is,

$$\tau = \tau + \beta. \quad (\text{A.3})$$

After a time  $\tau = \beta$  the particle returns to its initial position, being  $\beta$  related to a finite temperature.

Applying these steps to equation (A.1), we get

$$\begin{aligned} Z[\beta] &= \text{tr}(e^{-\beta H}) = \int dq_i \langle q_i | e^{-\beta H} | q_i \rangle \\ &= \int_{q_i(0)}^{q_i(\beta)} e^{iS}. \end{aligned} \quad (\text{A.4})$$

This implicates that we have to integrate over all trajectories with the requirement that  $q(0) = q(\beta)$ . Equation (A.4) is the quantum partition function.

## B APPENDIX B: TOPOLOGY OF THE CHERN-SIMONS ACTION AND GROUND STATE DEGENERACY

A quantum field theory described in a flat spacetime with the Minkowski metric  $\eta_{\mu\nu}$  is given by

$$S = \int d^D x \mathcal{L}(\eta_{\mu\nu}, \phi, \partial_\mu \phi), \quad (\text{B.1})$$

being  $\mathcal{L}$  the Lagrangian density. We can easily go to a curved spacetime by changing this metric for the Einstein metric  $g^{\mu\nu}$  including a factor of  $\sqrt{-g} = \det g^{\mu\nu}$  [13]. The action becomes

$$S = \int d^D x \sqrt{-g} \mathcal{L}(g_{\mu\nu}, \phi, \partial_\mu \phi). \quad (\text{B.2})$$

The quantity  $d^D x \sqrt{-g}$  is invariant under general coordinate transformations [46], which makes the action also invariant.

Now let us take a look to the Chern-Simons action in equation (5.48). It does not depend on any metric whatsoever. The indexes from the fields are contracted with the Levi-Civita tensor. Given a general coordinate transformation  $x \rightarrow x'$ , a vector component  $A_\mu(x)$  transforms as

$$A_\mu(x) = \frac{\partial x'^\lambda}{\partial x^\mu} A'_\lambda(x'). \quad (\text{B.3})$$

For three general vector fields  $A(x), B(x)$  and  $C(x)$ , the transformation becomes

$$\begin{aligned} \epsilon^{\mu\nu\rho} A_\mu(x) B_\nu(x) C_\rho(x) &= \epsilon^{\mu\nu\rho} \frac{\partial x'^\sigma}{\partial x^\mu} \frac{\partial x'^\lambda}{\partial x^\nu} \frac{\partial x'^\tau}{\partial x^\rho} A'_\sigma(x') B'_\lambda(x') C'_\tau(x') \\ &= \det \left( \frac{\partial x'}{\partial x} \right) \epsilon^{\sigma\lambda\tau} A'_\sigma(x') B'_\lambda(x') C'_\tau(x'). \end{aligned} \quad (\text{B.4})$$

We also know that  $d^3 x = d^3 x' \det \left( \frac{\partial x}{\partial x'} \right)$ . Then,

$$d^3 x \epsilon^{\mu\nu\rho} A_\mu(x) B_\nu(x) C_\rho(x) = d^3 x' \epsilon^{\mu\nu\rho} A'_\mu(x') B'_\nu(x') C'_\rho(x'), \quad (\text{B.5})$$

which is invariant without the  $\sqrt{-g}$  term. The full calculation can be found in section I.10 from reference [13]. The Chern-Simons theory only depends on the topology of spacetime. In other words, the functional  $\int_M D\alpha e^{iS(\alpha)}$  only depends on the topology of the manifold  $M$  and not on the metric.

Which is the Hamiltonian related to the Chern-Simons action? The Hamilto-

nian is defined to be the temporal component of the energy-momentum tensor  $T$ , given by

$$T^{\mu\nu} = \frac{\delta S_{CS}}{\delta g_{\mu\nu}}. \quad (\text{B.6})$$

As the action does not depend on the metric, the energy tensor and so the Hamiltonian are both equal to zero. This implies in a vanishing energy spectrum, that is  $H = 0$ , typical from topological theories [47]. The zero energy Hamiltonian gives us a difficult task. How many eigenstates have zero energy? Consider the Chern-Simons theory on a torus in two dimensions, represented by Figure B.1.

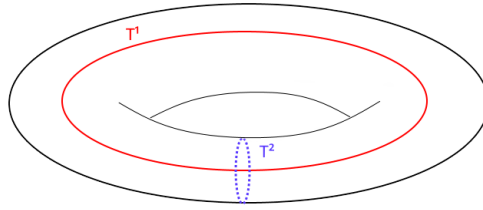


Figure B.1: Schematic representation of a Torus.  $T^1$  and  $T^2$  represents the possible curves of a closed integral on this surface.

We focus only on the Chern-Simons term, so the action in (5.63) becomes

$$S_{CS} = \int d^3x \frac{m}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda. \quad (\text{B.7})$$

The equation of motion for  $a_0$  is

$$\partial_1 a_2 - \partial_2 a_1 = 0. \quad (\text{B.8})$$

The simplest solution for this difference equation is letting  $a_i$  to be a constant. We need to find a gauge invariant object to parameterize these solutions. The simplest way is to integrate it, that is

$$w_i = \oint_{T_i} dx^j a_j, \quad (\text{B.9})$$

being  $i = 1, 2$  and  $j = 1, 2$ . Considering that the curves  $T^1$  and  $T^2$  has radius  $R_1$  and  $R_2$  respectively, a gauge transformation of  $a_j \rightarrow a_j + \frac{1}{R_j}$  implies that  $w_j \rightarrow w_j + 2\pi$ . This is the same problem we had in section 5.6.1, and so we can use the same argument to solve it.

We define a new operator called Wilson Loop, given by

$$W_i = \exp\left(i \oint_{T_i} dx^j a_j\right) = \exp(iw_i). \quad (\text{B.10})$$

These operators are observables of the theory and are essentially phase factors [48]. We impose

canonical commutation rules to the field components, so that

$$[a_1(x), a_2(x')] = \frac{2\pi i}{m} \delta^2(x - x'), \text{ and} \quad (\text{B.11})$$

$$[w_1, w_2] = \frac{2\pi i}{m}. \quad (\text{B.12})$$

The commutation relations implies that

$$W_1 W_2 = e^{\frac{2\pi i}{m}} W_2 W_1. \quad (\text{B.13})$$

This algebra is pretty much similar to the magnetic algebra described previously. We can interpret the Wilson Loop as a charged particle to propagate around the torus either by the surface  $T^1$  or  $T^2$ . We see that there is a difference in the path taken. The algebra in equation (B.13) can not be implemented in a single vacuum state [8], the smallest representation in the Hilbert space has dimension  $m$ , and so the ground state has a  $m$  degeneracy. For a general surface of the genus  $g$ , the degeneracy would be  $m^g$  [9], which means that the ground state degeneracy depends on the topology of the space.

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