Universidade Estadual de Londrina

RIELI TAINÁ GOMES DOS SANTOS

CHERN-SIMONS:

TEORIA DE BORDA COM FORMULAÇÃO DA MATRIZ *K*

Londrina
2025

RIELI TAINÁ GOMES DOS SANTOS

CHERN-SIMONS:

TEORIA DE BORDA COM FORMULAÇÃO DA MATRIZ *K*

Dissertação apresentada ao Programa de Mestrado em Física da Universidade Esta- dual de Londrina para obtenção do título de Mestre em Física.

Orientador: Prof. Dr. Pedro Rogério Sergi Gomes.

Londrina
2025

Ficha de identificação da obra elaborada pelo autor, através do Programa de Geração Automática do Sistema de Bibliotecas da UEL

R555 Santos, Rieli Tainá Gomes dos.
Chern-Simons : edge theory with K-matrix formulation / Rieli Tainá Gomes dos Santos. - Londrina, 2025.
98 f.

Orientador: Pedro Rogério Sergi Gomes.
Dissertação (Mestrado em Física) - Universidade Estadual de Londrina, Centro de Ciências Exatas, Programa de Pós-Graduação em Física, 2025.
Inclui bibliografia.

1. U(1)^N Chern-Simons - Tese. 2. K-matrix formulation - Tese. 3. Lagrangian subgroup - Tese. 4. Parafermions - Tese. I. Gomes, Pedro Rogério Sergi. II. Universidade Estadual de Londrina. Centro de Ciências Exatas. Programa de Pós-Graduação em Física. III. Título.

CDU 53

RIELI TAINÁ GOMES DOS SANTOS

CHERN-SIMONS:
TEORIA DE BORDA COM FORMULAÇÃO DA MATRIZ *K*

Dissertação apresentada ao Programa de Mestrado em Física da Universidade Esta- dual de Londrina para obtenção do título de Mestre em Física.

BANCA EXAMINADORA

____
Orientador: Prof. Dr. Pedro Rogério Sergi Gomes
Universidade Estadual de Londrina - UEL

____
Profa. Dra. Paula Fernanda Bienzobas
Universidade Estadual de Londrina - UEL

____
Prof. Dr. Heitor Casasola Cinel
Universidade Estadual de Londrina - UEL

Londrina, 04 de setembro de 2025.

To my grandmother, Neonila Demczuk Gomes.

AGRADECIMENTOS

Eu obtive um crescimento pessoal e acadêmico muito grande durante meu mestrado e várias pessoas em minha vida contribuíram para o meu crescimento. Eu agradeço imensamente ao meu orientador, professor Pedro Gomes, o qual me orientou da melhor forma possível e me motivou a ir além do formalismo matemático, despertando em mim a curiosidade de compreender os fundamentos físicos dos fenômenos e de questionar aquilo que eu não entendia. Além disso, agradeço aos conselhos acadêmicos e pessoais que recebi e pelo suporte recebido em toda a pesquisa. Agradeço também à professora Paula Bienzobás pelas várias discussões interessantes sobre a Física e por todas as conversas, apoio e conselhos. Sinto-me muito grata de ter tido professores com imenso conhecimento e empatia durante o mestrado.

Um agradecimento especial ao meu noivo, Arthur L. Medeiros, o qual me forneceu suporte durante todo o mestrado e me incentivou a admirar os pequenos detalhes na vida e aproveitar a jornada mesmo com dificuldades. Agradeço pelas conversas agradáveis e pelos risos em meio a todas as responsabilidades.

Agradeço ao meu pai e à minha irmã pelo apoio emocional nos momentos de dificuldade e por sempre me motivarem a seguir meus sonhos. Agradeço também às pessoas incríveis que conheci no Canadá durante minha estadia, que se tornaram uma família para mim e me ajudaram em todos meus momentos morando no Canadá, à Ilda, Diane, Ray e Benjamin.

Agradeço a todos os meus familiares que me acompanharam durante meu mestrado e aos demais professores que participaram da minha trajetória. Um agradecimentos, também, às minhas amizades, antigas e novas.

Por fim, agradeço à Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) pelo apoio financeiro durante o mestrado.

"Let us choose for ourselves our path in life,
and let us try to strew that path with
flowers."
Émilie Du Châtelet

RESUMO

SANTOS, R. T. G.. **Chern-Simons:** teoria de borda com formulação da matriz K. 2025. 97 f. Dissertação (Mestrado em Física) – Universidade Estadual de Londrina, Londrina, 2025.

Esta dissertação tem como objetivo estudar a gapeabilidade da teoria de borda de teorias de Chern-Simons $U(1)^N$ utilizando a formulação da matriz K. A formulação da matriz K é usada para descrever *qualquer* estado topológico abeliano em diferentes frações de preenchimento. Neste trabalho, revisamos algumas propriedades interessantes que surgem na teoria do bulk de uma teoria de Chern-Simons $U(1)^N$, como a equivalência topológica entre teorias com K-matrizes distintas, simetrias anyonicas e equivalência estável, com exemplos para cada uma dessas propriedades. Em seguida, analisamos as condições sob as quais a borda de uma teoria de Chern-Simons multicomponente pode ser gapeada. A primeira condição é que a carga central da teoria seja nula, o que significa que a borda tem o mesmo número de modos contrapropagantes. Essa condição por si só não garante que a borda possa ser gapeada. O requisito adicional para a gapeabilidade é a validade da condição nula de Haldane, que pode ser formalizada em termos de subgrupos Lagrangianos. Se a teoria de borda tiver pelo menos um subgrupo Lagrangiano e a carga central for zero, a borda pode ser gapeada por termos de perturbação de cosseno que condensam partículas na borda. Ademais, são apresentados exemplos sobre a gapeabilidade das teorias em sistemas com preenchimentos $\nu = 8/9$, que podem ser gapeados, e $\nu = 2/3$, em que a borda é protegida, seguidos pelo caso geral em que $K = \mathrm{diag}(k_1, -k_2)$, onde a teoria de borda pode ser gapeada se k_1k_2 for um quadrado perfeito. Além disso, apresentamos que o estado com $\nu = 8/9$ possui Z_3 parafermions na extremidade da interface formada pelas fases $\nu = 1$ e $\nu = -1/9$ e $K = \mathrm{diag}(k_1, k_2) = \mathrm{diag}(n^2, -m^2)$ possui Z_{mn} parafermions ligados às extremidades da interface, em acordo com os resultados de Ref. [1].

Palavras-chave: Teoria de borda; $U(1)^N$ Chern-Simons; Formulação da matriz K; Subgrupo Lagrangiano; Parafermions.

ABSTRACT

SANTOS, R. T. G. **Chern-Simons:** Edge theory with K-matrix formulation. 2025. 97 p. Master's Thesis (Master in Physics) – State University of Londrina, Londrina, 2025.

This dissertation aims to study the gappability of the edge theory of $U(1)^N$ Chern-Simons theories using the K-matrix formulation. The K-matrix formulation is used to describe *any* abelian topological state at different filling fractions. We review some interesting properties that arise in the bulk of a $U(1)^N$ Chern-Simons theory, such as the topological equivalence between theories with distinct K-matrices, anyonic symmetries, and stable equivalence, with examples for each of these properties. Subsequently, we analyze the conditions under which the edge of a multi-component Chern-Simons theory can be gapped. The first condition is that the central charge of the theory is null, which means that the edge has the same number of counter-propagating modes. This condition alone does not guarantee that the edge can be gapped. The additional requirement for gappability is the validity of Haldane's null condition, which can be formalized in terms of Lagrangian subgroups. If the edge theory has at least one Lagrangian subgroup in the system and the central charge is zero, the edge can be gapped by cosine perturbation terms that condense particles at the edge. Examples of the gappability of the states at filling $\nu = 8/9$, which can be gapped, and $\nu = 2/3$, in which the edge is protected, are presented, followed by the general case where $K = \text{diag}(k_1, -k_2)$, where the edge theory can be gapped if $k_1 k_2$ is a perfect square. In addition, we presented that the state with $\nu = 8/9$ has Z_3 parafermions at the endpoint of the interface formed by the phases $\nu = 1$ and $\nu = -1/9$ and $K = \text{diag}(k_1, k_2) = \text{diag}(n^2, -m^2)$ has Z_{mn} parafermions bound to the endpoints of the interface, in agreement with the results of Ref. [1].

Key-words: Edge theory; $U(1)^N$ Chern-Simons; K-matrix formulation; Lagrangian subgroup; Parafermions.

LISTA DE FIGURAS

LISTA DE ABREVIATURAS E SIGLAS

CS	Chern-Simons
TO	Topological Order
FQH	Fractional Quantum Hall
QHE	Quantum Hall Effect
EoM	Equation of Motion
K-M	Kac-Moody

SUMÁRIO

1 INTRODUCTION

In recent years, the discovery of topological quantum phases of matter has aroused great interest in Condensed Matter. The importance of these phases of matter continues to make them an active field of research to this day, due to their numerous applications in quantum computing, as well as in the fields of insulators and superconductors. Topological phases are distinguished by their topological features rather than by their symmetries, and cannot be characterized by the usual Landau paradigm of spontaneous symmetry-breaking. The Landau paradigm [6] is related to the symmetry breaking of ordinary symmetries, which are characterized by local order parameters that are charged under these symmetries. Generalized symmetries extend the Landau paradigm to include the concept of Topological Order (TO), which can be understood as spontaneous symmetry breaking of finite higher-form symmetries. A review of TO can be found in [7–12].

Systems with TO are characterized by the presence of quasiparticles with fractional or non-Abelian statistics and topological ground-state degeneracy, which depend on the topology of the space. The ground-state degeneracy exhibits robustness because local perturbations cannot remove it. The most commonly observed topological states experimentally are the fractional quantum Hall (FQH) states. A review of FQH states can be found in [2, 9, 13–21].

The $U(1)$ Chern-Simons (CS) theory is an effective topological quantum field theory that captures the behavior of certain topologically ordered phases, e.g., the FQH states [3, 9, 11, 12, 17, 18, 20–27]. This system is described in terms of an emergent gauge field a_μ, and the pure theory does not have any dynamical degrees of freedom. The $U(1)$ CS theory is topological because its action does not depend on the spacetime metric, and the invariant objects are global quantities related to the topology of the space. For example, the ground-state degeneracy in compact manifolds depends on the topology of the space, which is a global quantity. Furthermore, the observable quantities of this theory are non-local operators, meaning that the quasiparticles exhibit non-local features, such as non-trivial statistics.

The $U(1)$ CS theory can describe the low-energy behavior of quantum Hall states with filling fraction $\nu = 1/\kappa$, where κ is the level of the theory. These phases can host abelian quasiparticles (anyons) with fractional statistics and fractional electric charges. The states with filling fraction $\nu = 1/\kappa$, where κ is an odd integer, are the well-known Laughlin states [28]. The article [19] from 2021 presents a review of the literature on experimental techniques used to detect quasiparticles with fractional statistics.

The $U(1)$ Chern-Simons theory can be analyzed from the perspective of higher-form symmetries within the context of generalized symmetries. Owing to the non-commutativity of the Wilson lines, the $U(1)$ theory has a discrete 1-form symmetry $\mathbb{Z}_\kappa$ related to the level κ of the theory [11, 24].

In the $U(1)$ Chern-Simons theory, the charged objects under the generalized symmetry are Wilson operators, which are non-local gauge-invariant objects that describe the world line of quasiparticles. The non-locality of the Wilson lines is a feature stemming from the topological aspects of the theory. When two indistinguishable quasiparticles are exchanged, the wave-function can acquire a fractional phase that differs from that of ordinary particles (fermions and bosons). This is the Aharonov-Bohm effect, which can be analyzed by studying the algebra of Wilson lines in the theory. The phase acquired in this process does not depend on the distance between the exchanged quasiparticles, which is another topological aspect of this theory related to long-range entanglement.

The quantum Hall effect (QHE) arises naturally in CS theory when the gauge fields are minimally coupled to the electromagnetic field A_μ. The equation of motion considering the coupling shows that the electric current in the x^i direction induces a current in the x^j direction, which is the essence of the QHE. In contrast, the fractional electric charge of the quasiparticles is obtained by coupling the theory with a matter current. In addition, within the context of coupling with a matter current, it is possible to infer the flux-charge relation, where each quasiparticle has a thin magnetic flux attached to it.

The $U(1)$ bulk theory is defined in a manifold $\mathcal{M}$ and is gauge-invariant if the theory has no boundary or if the boundary term can be neglected. However, in the presence of a boundary $\partial\mathcal{M}$, the bulk theory has a gauge anomaly that can be canceled only if we consider the dynamical gapless excitations at the edge. In this case, the theory at the edge is non-trivial with the presence of chiral gapless edge modes. In the case of $U(1)$ CS theory, there is a unique edge mode with one-way propagation that is stable against perturbations. Therefore, it is not possible to open a gap in the edge theory, and the edge mode remains gapless. Thus, in the presence of a boundary, the system hosts *local* degrees of freedom, differently from the bulk physics, where the observables of the theory are non-local.

The FQH states with hierarchical levels can be described by a $U(1)^N$ CS effective topological field theory, where the system has N emergent gauge fields a^I_μ, with $I = 1, 2, \ldots, N$. Hierarchical states are a generalization of the single-component case and can be studied using the K-matrix formulation, proposed by Wen and Zee in [29]. In fact, not only hierarchical states but also any abelian topological phase can be studied using the K-matrix formulation, which is a compact and elegant way of describing more general topologically ordered phases. Due to TO, this model has quasiparticles with non-trivial braiding and topological invariants, such as Hall conductivity and topological ground-state degeneracy, which depend on the K-matrix. The quasiparticles in the system are defined modulo local particles, and the number of distinct particles in the bulk theory is the same as the ground-state degeneracy on the torus; that is, the number of distinct quasiparticles is obtained from the determinant of the K-matrix.

The $U(1)^N$ bulk theory can be studied as a generalization of the single-component $U(1)$ Chern-Simons theory with a similar construction, and both systems share physical features. For example, the coupling of the gauge fields a^I_μ with an electromagnetic field A_μ through a charge vector t defines induced currents orthogonal to the electric field, similar to the single-component case. In addition, the coupling with a matter current through the quasiparticle vector l gives rise to a charge-flux relation and fractional electric charge for the quasiparticles. However, in the multi-component CS theory, there are new interesting properties, such as equivalence between theories [30], the concept of anyonic symmetries [5, 30–32], and stable equivalence [5, 30, 33, 34].

The equivalence between topological phases is related to a basis change in the gauge field through a transformation matrix $G \in GL(N, \mathbb{Z})$ (which also changes the vector charge t and the quasiparticle vector l), leading to a new theory containing the same topological features as the original. In this case, two different K-matrices can represent the same topological phase with the same set of quasiparticles, although they are labeled in distinct ways.

In addition, anyonic symmetries are operations that permute anyons, preserving the braiding exchange phase and fusion rule of the quasiparticles. Anyonic symmetries of the system can be found using the K matrix and a matrix transformation that leaves the Lagrangian invariant. However, in some cases, the K-matrix itself does not account for all realizable anyonic symmetries, and we need to enlarge K through trivial sectors to determine the remaining anyonic symmetries. This process of enlarging the K-matrix using trivial sectors is called stable equivalence.

The edge theory of a $U(1)^N$ CS theory will also have dynamical gapless mode excitations because the bulk action is gauge-invariant up to boundary terms. However, unlike the single-component case, this system can have more than one edge mode propagating at its edge. The edge modes are chiral and can move to the left or right. In the particular case where the theory has an equal number of modes propagating to the right and left (no net chirality), the edge theory has the potential to be gapped by a perturbation term [29, 30, 33–57].

However, the gappability of non-chiral edges is not guaranteed. There are states without topological order and with short-range entanglement in the bulk of the system. These phases are "trivial" in the bulk with a non-trivial edge theory. The edge modes are gapless and protected by symmetry, and the edge can only be gapped if (spontaneous or explicit) symmetry breaking occurs. Therefore, in the absence of symmetry breaking, the edge is robust against perturbations, and the edge excitations cannot be localized because of the protection provided by the symmetry. These edge states are called Symmetry Protected Phases (SPT's) and are widely studied [34, 39, 46, 52, 58–72]. Therefore, if symmetry forbids perturbation Higgs terms, the system has SPT phases with non-trivial edge mode excitations. The edges defined in two- or higher-dimensional manifolds can also be gapped with intrinsic topological order.

The gappability of edge states without net chirality can be studied using the concept of Lagrangian subgroups [41, 43, 47, 48, 72–75], which are subgroups containing topological quasiparticles with particular properties that are condensed at the gapped edge. The edge modes can be gapped when the edge theory has at least one Lagrangian subgroup. Therefore, this process can also be understood from the perspective of anyon condensation at the edge [41, 43, 44, 53, 55, 74–77], where the particles in the Lagrangian subgroup are condensed. At the same time, the other quasiparticles that are not in the Lagrangian subgroup are confined at the edge.

The interaction between edge modes at the interfaces can also be studied. This interaction occurs when two different edges are in proximity, and the edge modes can tunnel from one edge to the other [33, 78]. The same process occurs when the bulk is thin enough to allow interaction between the top and bottom edges in a quantum Hall state.

In addition, the $U(1)^N$ edge theory can host more exotic anyons at interfaces that obey non-Abelian braiding statistics[1] [37, 40, 42–44, 70, 79–87]. These non-Abelian quasiparticles are of great interest because they have wide potential applications in fault-tolerant topological quantum computing [88, 89] due to their topological ground-state degeneracy.

The bulk-edge correspondence in the Chern-Simons approach provides a method for identifying all edge excitations with their bulk counterparts. The bulk theory is defined in a manifold $\mathcal{M}$, and when this manifold has a boundary $\partial\mathcal{M}$, chiral boson fields capture the gapless excitations in the edge theory. When there is an equal number of left and right propagation modes in the edge (no net chirality), the theory can be gapped out through perturbation processes, as backscattering or superconductor terms [29, 30, 33–35, 37–54, 56, 57].

This dissertation aims to analyze in detail the $U(1)^N$ Chern-Simons edge theory by studying perturbations at the edge and the conditions for gappability using the K-matrix formulation. A literature review of the edge theory of the $U(1)^N$ Chern-Simons theory was conducted, and the main results are summarized in this dissertation.

This dissertation is structured as follows. First, in Chapter 2, the single-component $U(1)$ Chern-Simons theory is presented, along with its main characteristics, such as quasiparticles with electric fractional charges and fractional statistics. Then, the edge theory

[1] In contrast to the case of abelian anyons, exchanging non-Abelian anyons gives more than just a phase. For example, it can also rotate the state to a different wavefunction.

of the $U(1)$ CS theory is presented in Chapter 3, where there is a boson chiral field propagating through the edge. The edge theory is robust against perturbations; therefore, the modes cannot be gapped and remain gapless.

In Chapter 4, the multi-component $U(1)^N$ bulk CS theory is studied using the K-matrix formulation. The main features of the system are analyzed, and some interesting properties are presented, such as the equivalence between topological phases, anyonic symmetries, and stable equivalence.

In addition, we present system examples where a topological phase in contact with an s-wave superconductor is topologically equivalent to a bosonic state at a different filling fraction. This bosonic state associated with a K-matrix has three anyonic symmetries, which will be studied in detail in the following. We will show that the transformation G associated with the K-matrix realizes only one of the three anyonic symmetries of the system. Thus, in the last example, we will use stable equivalence to enlarge the K-matrix by a trivial sector to realize the two remaining anyonic symmetries.

Finally, in Chapter 5, we study in detail the edge theory of multi-component $U(1)^N$ Chern-Simons theory without symmetries. We present that the gappability condition can be derived from the existence of at least one Lagrangian subgroup in the edge theory, utilizing the concept of anyon condensation as an interpretation of the gapped edge. Three examples will be addressed concerning the states with $\nu = 8/9$, which can be gapped; $\nu = 3/2$, which cannot be gapped, and the edge modes remain gapless; and the general case where $K = \text{diag}(k_1, -k_2)$, which can be gapped if $k_1 k_2$ is a perfect square.

In addition, in the last chapter, we also show that the edge theories studied in this dissertation can host quasiparticles with non-Abelian statistics when we interpret each topological phase as an interface that separates two different phases; for example, the system with filling $\nu = 8/9$ can be interpreted as an interface separating the phases $\nu = 1$ and $\nu = -1/9$. These non-Abelian quasiparticles are bound to the endpoints of this interface.

2 $U(1)$ CHERN-SIMONS THEORY

In this chapter, Abelian $U(1)$ Chern-Simons (CS) theory will be studied in detail. The $U(1)$ Chern-Simons theory is a single-component topological effective field theory. This theory possesses several interesting properties, such as the capacity to generate Abelian anyons[1] with fractional statistics and charge. The CS theory is defined in manifolds of odd space-time dimensions. In this work, the focus will be on the theory defined on a (2+1)-dimensional spatial plane. However, generalizing it to higher dimensions is not difficult to implement.

2.1 Bulk Lagrangian

We will begin studying the bulk theory of this spatial plane, which is described by a Chern-Simons term. The plane will be considered infinitely spaced without boundaries in this chapter.

The bulk Chern-Simons action in (2+1) dimensions is defined as

$$S[a] = \int_{\mathcal{M}} \frac{\kappa}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \tag{2.1}$$

where κ is the level of the CS theory and a_μ is an $U(1)_a$ emergent gauge field. This action is first-order in space-time derivatives; therefore, this term dominates at long distances (low-energy regime) compared to the Maxwell term, which is second-order in derivatives. Further, because the Levi-Civita tensor $\varepsilon^{\mu\nu\rho}$ is completely antisymmetric in its indices, the CS action is Lorentz invariant.

The Chern-Simons theory is topological since its action does not depend on the metric $g_{\mu\nu}$ when placed in a curved space. Therefore, its invariants are global quantities related to the topology of the space, and the CS action is invariant under smooth deformations of the geometry of the space. The stress tensor, which is explicitly dependent on the metric, vanishes. As a result, the Hamiltonian of this system is zero, which shows that the pure Chern-Simons theory lacks local dynamical degrees of freedom.

Notice that the CS action is gauge-invariant up to boundary terms. Since we are considering that the manifold $\mathcal{M}$ does not have a boundary, the CS term is indeed gauge-invariant. In the next chapter, we will analyze the case where $\mathcal{M}$ has an edge and the boundary terms cannot be neglected.

---

[1] The name "anyon" comes from particles obeying *any* statistics, that is, these particles do not obey either bosonic or fermionic statistics.

The equations of motion (EoM) can be derived from the CS Lagrangian using

$$\partial_\beta \frac{\partial \mathcal{L}_{CS}}{\partial(\partial_\beta\, a_\alpha)} - \frac{\partial \mathcal{L}_{CS}}{\partial a_\alpha} = 0, \tag{2.2}$$

as

$$\frac{\kappa}{4\pi}\varepsilon^{\alpha\mu\nu} f_{\mu\nu} = 0 \implies f_{\mu\nu} = 0, \tag{2.3}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength tensor. This equation of motion shows that the system does not have local propagating degrees of freedom.

The solution to the EoM represented in equation 2.3 is a pure gauge configuration

$$a_\mu = \partial_\mu \lambda,$$

for some arbitrary scalar field λ. This type of solution describes flat connections, since $f = da$ is the curvature of this connection. In other words, when $f = 0$, the curvature associated with the connection is zero. It is possible to see that this system will not have local propagating excitations, since the flat connections do not carry energy or momentum, unlike Maxwell's free theory, in which the solutions are propagating plane waves describing connections with curvature ($f \neq 0$).

It is possible to define a 2-form topological current associated with the CS theory as

$$J^{\mu\nu} = \varepsilon^{\mu\nu\rho} a_\rho, \tag{2.4}$$

$$\partial_\mu J^{\mu\nu} = \varepsilon^{\mu\nu\rho}\partial_\mu a_\rho = 0, \tag{2.5}$$

which is conserved because of the EoM in equation 2.3. Therefore, the CS theory has a 1-form symmetry with charge

$$Q(\Sigma_1) = \int_{\Sigma_1} *J, \tag{2.6}$$

which is defined on a one-dimensional closed submanifold Σ_1 and it is a conserved quantity. Using

$$\begin{aligned} *J &= \frac{1}{2} J^{\mu\nu}\varepsilon_{\mu\nu\rho} dx^\rho \\ &= a_\rho dx^\rho, \end{aligned} \tag{2.7}$$

the charge in equation 2.6 becomes

$$Q(\Sigma_1) = \int_{\Sigma_1} a. \tag{2.8}$$

The charged objects under the 1-form symmetry are extended objects with support along a line, the Wilson line operators. These operators are the gauge-invariant observables of the CS theory, describing the worldline of the quasiparticles in the system. They are non-local objects since the topological phase is robust to any local perturbation, and they are defined as

$$W_n = \exp\left(in \oint_C a\right), \tag{2.9}$$

where C must be a closed curve (or infinitely long) in space to assure the gauge-invariance of this object.

The κ level of the CS theory is quantized through large gauge transformations. To see this, it is convenient to write the CS action in equation 2.1 as

$$S[a] = \int d^3x \frac{\kappa}{4\pi} a_0 \varepsilon^{ij} f_{ij} + \dots. \tag{2.10}$$

Consider the action defined in a manifold $\mathcal{M} = \mathcal{S}^1 \times \mathcal{S}^2$, where the time component is defined in a circle $\mathcal{S}^1$ of circumference L_0. In addition, define a flux due to the presence of a Dirac monopole inside $\mathcal{S}^2$ as

$$\frac{1}{2\pi} \int_{\mathcal{S}^2} f \in \mathbb{Z}. \tag{2.11}$$

Assuming that the large gauge transformation winds around the time direction as

$$a_0 \to a_0 + \partial_0 \lambda = a_0 + \frac{2\pi}{L_0}, \tag{2.12}$$

with $\lambda = 2\pi x_0 / L_0$, then this transformation results in the compactness of the gauge field a_0.

The action variation under this large gauge transformation is given by

$$\delta S[a] = \int_{\mathcal{M}} d^3x \frac{\kappa}{4\pi} \frac{2\pi}{L_0} \varepsilon^{ij} f_{ij} = 2\pi\kappa \int_0^{L_0} dx^0 \frac{1}{L_0} \left(\frac{1}{2\pi} \int_{\mathcal{S}^2} \frac{1}{2} \varepsilon^{ij} f_{ij} \right), \tag{2.13}$$

where the bracket term is identified as the flux in equation 2.11, which is an integer value. Therefore,

$$\delta S[a] = 2\pi\kappa\mathbb{Z}. \tag{2.14}$$

For the CS quantum theory to be invariant under this large gauge transformation, it is necessary that $e^{i\delta S[a]} = 1$ so that the path integral is well-defined[2]. Thus, to satisfy this condition for any integer $\mathbb{Z}$, the level of the theory $\kappa \in \mathbb{Z}$ is quantized in integer values.

2.2 Coupling with Eletromagnetic Field

It is possible to couple the CS term with a $U(1)_A$ background electromagnetic field A_μ through a minimal coupling as

$$S[a] = \int_{\mathcal{M}} \frac{\kappa}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - A_\mu J^\mu, \tag{2.15}$$

[2] The invariance of the path integral is enough since the observables of the theory are obtained from the path integral.

with topological gauge-invariant current

$$J^{\mu} = \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} \partial_{\nu} a_{\rho}, \tag{2.16}$$

where the normalization constant was chosen to maintain the coupling term invariant under large gauge transformations. Then, the action becomes

$$S[a] = \int_{\mathcal{M}} \frac{\kappa}{4\pi} \varepsilon^{\mu\nu\rho} a_{\mu} \partial_{\nu} a_{\rho} - \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} A_{\mu} \partial_{\nu} a_{\rho}. \tag{2.17}$$

Let us use the Euler-Lagrange equation in equation 2.2, considering the gauge field a_{μ} as a dynamical field, to obtain the equation of motion

$$\begin{aligned} \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} \partial_{\nu} a_{\rho} &= \frac{1}{\kappa} \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} \partial_{\nu} A_{\rho} \\ \implies a_{\mu} &= \frac{1}{\kappa} A_{\mu} \text{ (locally)}. \end{aligned} \tag{2.18}$$

Applying this EoM in the CS action in equation 2.17 to integrate the field a_{μ} out, one gets

$$S[A] = -\int_{\mathcal{M}} \frac{1}{4\pi\kappa} \varepsilon^{\mu\nu\rho} A_{\mu} \partial_{\nu} A_{\rho}, \tag{2.19}$$

which is the electromagnetic response action. Notice that this is a CS term for A_{μ} with fractional level $1/\kappa$.

The current response to the electromagnetic field is defined as

$$\langle J^{\mu}_{ind} \rangle = -\frac{\delta S[A]}{\delta A_{\mu}} = \frac{1}{2\pi\kappa} \varepsilon^{\mu\nu\rho} \partial_{\nu} A_{\rho}, \tag{2.20}$$

and, then, the spatial component of the electromagnetic current is given by

$$\begin{aligned} J^{i}_{ind} &= \frac{1}{2\pi\kappa} (\varepsilon^{ij0} \partial_j A_0 + \varepsilon^{i0j} \partial_0 A_j) \\ &= \frac{1}{2\pi\kappa} \varepsilon^{ij} (\partial_j A_0 - \partial_0 A_j) \\ &= \frac{1}{2\pi\kappa} \varepsilon^{ij} F_{j0} = -\frac{1}{2\pi\kappa} \varepsilon^{ij} F_{0j} \\ &= \frac{1}{2\pi\kappa} \varepsilon^{ij} E_j, \end{aligned} \tag{2.21}$$

with $F_{0j} = -E_j$. J^{i}_{ind} is the Hall current, which occurs when an electric field in the x^1 direction induces an electric current in the x^2 direction. Therefore, this equation indicates the Quantum Hall Effect.

The temporal component of the electromagnetic current is given by

$$\begin{aligned} J^{0}_{ind} = \rho^0 &= \frac{1}{2\pi\kappa} \varepsilon^{0ij} \partial_i A_j \\ &= \frac{1}{2\pi\kappa} B, \end{aligned} \tag{2.22}$$

with $B = \varepsilon^{ij}\partial_i A_j$. This equation is related to an excess density of electrons due to the variation of the magnetic field as $\delta n = \frac{1}{2\pi}\nu\delta B$. Therefore, the filling fraction associated with this system is given by $\nu = 1/\kappa$. Notice that when κ is an odd integer, this theory describes a Laughlin state. The coefficient in equations 2.21 and 2.22 is the Hall conductivity

$$\sigma_H = \frac{1}{2\pi\kappa} = \frac{1}{2\pi}\nu. \tag{2.23}$$

2.3 Coupling with a Matter Current

To specify the local particle excitations of the system, let us couple the CS theory with a matter current j^μ as

$$S[a] = \int_{\mathcal{M}} \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - a_\mu j^\mu, \tag{2.24}$$

with a quasiparticle current

$$j^\mu = \frac{q}{2\pi}\varepsilon^{\mu\nu\rho}\partial_\nu b_\rho, \tag{2.25}$$

where b_ρ is a new single-component gauge field.

Using the Euler-Lagrange equation represented in equation 2.2, the EoM for a_μ is given by

$$\begin{aligned}
&\frac{\kappa}{4\pi}\varepsilon^{\alpha\mu\beta}(\partial_\beta a_\mu - \partial_\mu a_\beta) = -j^\alpha \\
&\frac{\kappa}{4\pi}\varepsilon^{\alpha\beta\mu}f_{\beta\mu} = j^\alpha \\
&j^\mu = \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}f_{\nu\rho},
\end{aligned} \tag{2.26}$$

and because of the Bianchi identity,

$$\partial_\mu j^\mu = \partial_\mu\left(\frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}f_{\nu\rho}\right) = 0, \tag{2.27}$$

implying that the matter current is conserved since $f_{\mu\nu}$ is antisymmetric. This guarantees the gauge invariance of this term.

The spatial component of the matter current is

$$\begin{aligned}
j^i &= \frac{\kappa}{4\pi}(\varepsilon^{ij0}f_{j0} + \varepsilon^{i0j}f_{0j}) \\
&= \frac{\kappa}{2\pi}\varepsilon^{ij}f_{j0} \\
&= \frac{\kappa}{2\pi}\varepsilon^{ij}E_j,
\end{aligned} \tag{2.28}$$

with $F_{j0} = E_j$. Thus, as a result of this relation, we can infer that the emergence of an electric field is due to the presence of an electric current perpendicular to E.

On the other hand, the temporal component of the electromagnetic current is given by

$$\begin{aligned} j^0 = \rho^0 &= \frac{\kappa}{4\pi}\varepsilon^{0ij} f_{ij} \\ &= \frac{\kappa}{4\pi}(\varepsilon^{12} f_{12} + \varepsilon^{21} f_{21}) \\ &= \frac{\kappa}{2\pi} B, \end{aligned} \tag{2.29}$$

with $B = f_{12}$ and $\varepsilon^{12} = 1$. Through this equation, it is possible to realize that the physical meaning of coupling the gauge field a_μ with a matter current j^μ is to tie magnetic flux to charge. Notice that they are locally proportional to each other with the proportional constant given by the CS level κ. Because of that, equation 2.29 is called the charge-flux relation.

Consider N non-relativistic point particles in a 2-dimensional space with charge density given by

$$\rho^0(x) = q \sum_{a=1}^{N} \delta^2(\vec{x} - \vec{x}_a(t)), \tag{2.30}$$

and a quasiparticle current

$$\vec{j} = q \sum_{a=1}^{N} \dot{x}_a(t)\, \delta^2(\vec{x} - \vec{x}_a(t)), \tag{2.31}$$

where the a_{th}-quasiparticle is following a trajectory $\vec{x}_a(t)$. Due to the charge-flux relation in equation 2.29,

$$B = \frac{2\pi q}{\kappa} \sum_{a=1}^{N} \delta^2(\vec{x} - \vec{x}_a(t)), \tag{2.32}$$

which implies that each point particle has a magnetic flux with intensity $2\pi q/\kappa$ attached to it, as represented in Figure 1.

Notice that we can generalize equations 2.30 and 2.31 as

$$j^\mu(x^0, \vec{x}) = q \frac{dy^\mu(x^0)}{dx^0} \delta^2(\vec{x} - \vec{y}(x^0)). \tag{2.33}$$

Using equation 2.9 for the Wilson operator, notice that

$$\begin{aligned} W_q[C] &= e^{iq \oint_C a} \\ &= \exp\left(iq \int dy^\mu a_\mu\right) \\ &= \exp\left(iq \int dx^0 \frac{dy^\mu(x^0)}{dx^0} a_\mu\right) \\ &= \exp\left(iq \int dx^0 \frac{dy^\mu(x^0)}{dx^0} a_\mu\right) \\ &= \exp\left(iq \int d^2x \delta^2(\vec{x} - \vec{y}(x^0)) \int dx^0 \frac{dy^\mu(x^0)}{dx^0} a_\mu\right) \\ &= \exp\left(i \int d^3x\, j^\mu a_\mu\right), \end{aligned} \tag{2.34}$$

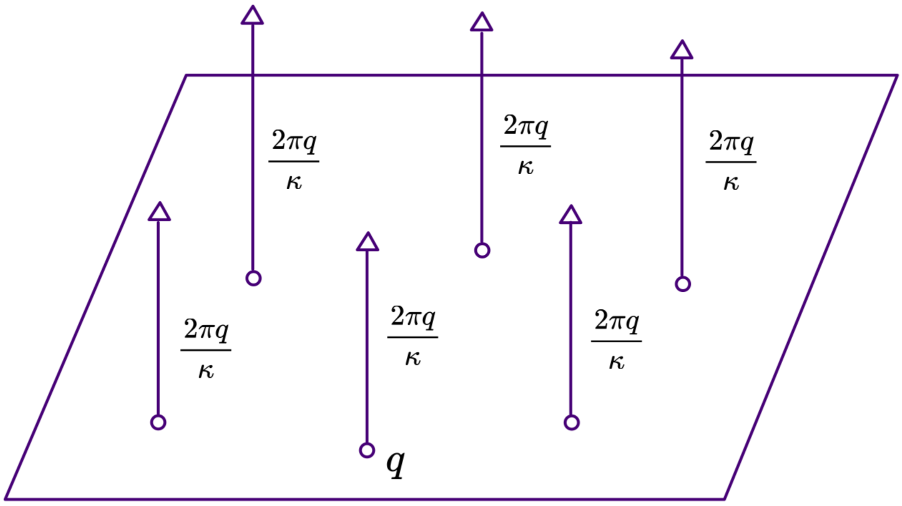


Figura 1 – A collection of quasiparticles with charge q with a magnetic flux of intensity $2\pi q/\kappa$ attached to them.

and, thus, the expectation value of the Wilson line operator

$$\langle W_q[C]\rangle = \int \mathcal{D}a \exp\left(iS[a] + i\int d^3x\, j^\mu a_\mu\right) \tag{2.35}$$

correspond to insert a coupling term $j^\mu a_\mu$ in the action. This term is parametrized by the current j^μ written in terms of a new gauge field b_μ to maintain the gauge invariance of the action. In other words, introducing a Wilson line at the action is equivalent to introducing an external charged particle.

The electric charge associated with this external probe particle can be calculated by coupling the CS theory in equation 2.24 to a background electromagnetic field A as

$$\begin{aligned} S[a] &= \int_{\mathcal{M}} \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - A_\mu J^\mu - a_\mu j^\mu \\ &= \int_{\mathcal{M}} \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - \frac{1}{2\pi}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho - \frac{q}{2\pi}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu b_\rho. \end{aligned} \tag{2.36}$$

The equation of motion for a_μ is obtained by using the Euler-Lagrange equation represented in equation 2.2 and it is given by

$$a_\mu = -\frac{1}{\kappa}A_\mu + \frac{q}{\kappa}b_\mu. \tag{2.37}$$

Substituting this EoM into the action in equation 2.36, one gets

$$S = \int d^3x\left(-\frac{q}{\kappa}A_\mu j^\mu + \dots\right), \tag{2.38}$$

which means that the Wilson line has an electric charge $Q = -q/\kappa$, which is, in general, fractional.

As the Wilson lines describe the worldline of quasiparticles, the excitations with fractional charge are quasiparticles in an FQH state. Therefore, the current matter j^μ can be interpreted as a current that describes the quasiparticles in the system. Notice that the quasiparticle has charge q under the a_μ field, but fractional electric charge $-q/\kappa$ under the electromagnetic field A_μ.

2.4 Quantization of Chern-Simons theory

To proceed to the quantization of the CS theory, let us write the Chern-Simons action in a canonical form as

$$S_{CS} = -\frac{\kappa}{4\pi}\int d^3x\, \varepsilon^{ij} a_i(\partial_0 a_j) + \dots \tag{2.39}$$

$$= \frac{\kappa}{2\pi}\int d^3x\, \partial_0 a_1 + \dots, \tag{2.40}$$

where the terms with spatial derivatives were omitted in $\dots$.

The canonical momentum can be obtained from the CS Lagrangian as

$$\Pi_1 = \frac{\partial \mathcal{L}}{\partial(\partial_0 a_1)} = \frac{\kappa}{2\pi} a_2. \tag{2.41}$$

By the other hand, the commutator between the gauge field and the canonical momentum is given by

$$[a_i(\vec{x}), \Pi_j(\vec{y})] = i\delta_{ij}\delta^2(\vec{x}-\vec{y}). \tag{2.42}$$

Using equation 2.41 in the expression above, one gets

$$[a_1(\vec{x}), a_2(\vec{y})] = \frac{2\pi i}{\kappa}\delta^2(\vec{x}-\vec{y}), \tag{2.43}$$

showing that the gauge fields a_1 and a_2 form a conjugate pair.

With the commutating relation in equation 2.43, it is possible to study the algebra of the following Wilson lines

$$W_1 = \exp\left(i\int dx^1 a_1\right)$$

$$W_2 = \exp\left(i\int dx^2 a_2\right),$$

as

$$W_1 W_2 = e^{-2\pi i/k} W_2 W_1, \tag{2.44}$$

or, more generally,

$$(W_1)^m (W_2)^n = e^{-2\pi i m n/k} (W_2)^n (W_1)^n. \tag{2.45}$$

Notice that the Wilson operators $(W_1)^k$ and $(W_2)^k$ behave like a transparent line in the sense that both commute with $(W_1)^m$ and $(W_2)^n$,

$$\begin{aligned}(W_1)^k(W_2)^m &= \exp\left(-\frac{2\pi ikm}{k}\right)(W_2)^m(W_1)^k \\ &= e^{-2\pi im}(W_2)^m(W_1)^k \\ &= (W_2)^m(W_1)^k,\end{aligned} \tag{2.46}$$

since $k \in \mathbb{Z}$ and $e^{-2\pi i\mathbb{Z}} = 1$. This means that the line

$$W^k[C] = \exp\left(ik\oint_C a\right) = (W[C])^k$$

does not induce a non-trivial holonomy, and it behaves as an identity by considering the $\mathbb{Z}_k$ algebra. Therefore, the operators associated with the 1-form symmetry belong to the $\mathbb{Z}_k$ group by performing a discrete symmetry $\mathbb{Z}_k^{(1)}$.

2.5 Anyon Statistics

In this section, we aim to analyze the non-trivial statistics of the quasiparticles that arise in our system when we adiabatically move one quasiparticle around another. We will obtain the phase acquired in this process through the interpretation of the Aharonov-Bohm effect and also through the algebra of the Wilson lines.

The charge-flux relation gives Aharonov-Bohm-type interactions. This interaction occurs when a quasiparticle with a_μ-charge q is adiabatically moved around a flux q/κ, as indicated in Figure 2. By the end of this process, the wavefunction associated with the quasiparticle q acquires a phase

$$e^{iq\Phi} = e^{i2\pi q^2/\kappa}. \tag{2.47}$$

This adiabatic process is interpreted as a double exchange of two identical particles, each one with a flux $\Phi = 2\pi q/\kappa$ attached to them. Therefore, the anyonic exchange phase arises when two identical particles are exchanged, that is, when a particle goes around another one at an angle of π, instead of 2π, as in the Aharonov-Bohm-type interaction. This process gives the anyonic phase of the particle q as

$$e^{i\frac{2\pi q^2}{2\kappa}} = e^{i\pi q^2/\kappa}. \tag{2.48}$$

The statistics θ is obtained through the phase $e^{\pm i\pi\theta}$ that the wavefunction acquires under exchange. Therefore, the statistics is given by

$$\theta = \frac{q^2}{\kappa}\text{mod } 2, \tag{2.49}$$

which is defined mod 2, since $e^{i2\pi} = 1$. Figure 3 shows four types of windings between two quasiparticles, where the statistics is represented in the first (counted from the left)

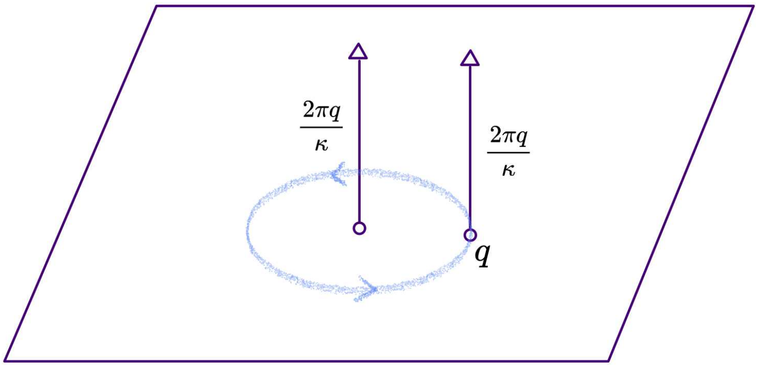


Figura 2 – A quasiparticle with a_μ charge q moving adiabatically around a flux $\Phi = 2\pi q/\kappa$.

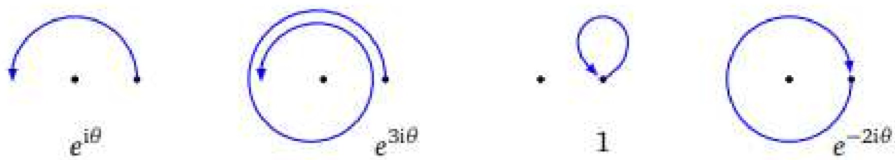


Figura 3 – Illustration of different windings between two quasiparticles, with the phase acquired in each process. From [2].

process, where this process represents an exchange up to a translation that does not change the acquired phase. The Aharonov-Bohm effect is represented in the last process.

Observe that when θ is an odd integer, $(e^{i\pi})^{\theta=odd} = -1$ and the particles respect fermionic statistics. By the other hand, when $(e^{i\pi})^{\theta=even} = 1$, the particles obey bosonic statistics. However, ν can assume fractional values. In this case, the particles obey *any* statistics. These are the anyons; quasiparticles with fractional statistics.

The same result for the statistics can be obtained through the Wilson lines, as they describe the world line of the quasiparticles in the system. Consider two Wilson lines W_1 and W_2, each one with charge q. The first Wilson operator is extended infinitely in the time direction, performing a path γ^0, and the second Wilson line encircles the first in a closed path γ^1, as indicated in Figure 4.

Notice that the anyonic phase does not change when we perform smooth deformations on the curve γ^1 that describes the trajectory of quasiparticle 2. What contributes to the phase is how many times the quasiparticle 2 winds around the flux attached to quasiparticle 1. Therefore, the phase is independent of the trajectory in which the quasiparticle

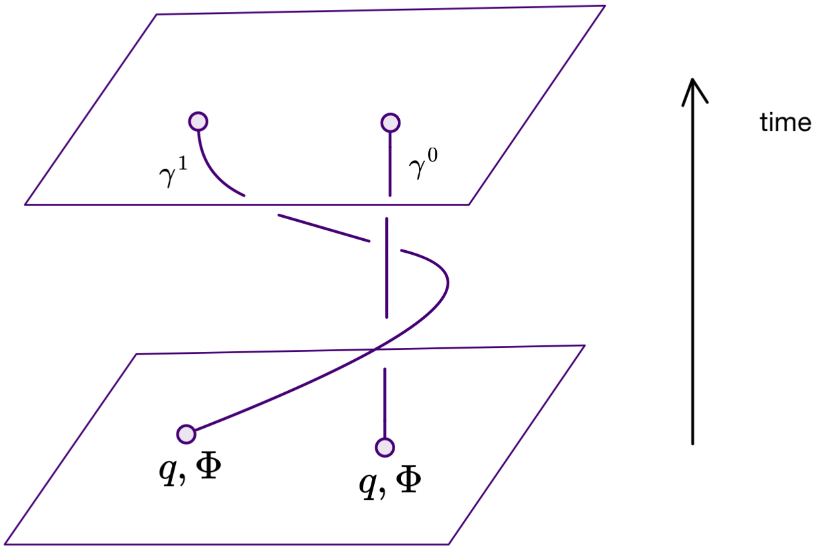


Figura 4 – A double exchange between two identical quasiparticles.

winds around the flux. In addition, notice that the winding process does not depend on the distance between quasiparticles. Therefore, the interaction between quasiparticles is also long-ranged.

Notice we can write equation 2.45 in terms of the linking number as

$$W_{\gamma^0}W_{\gamma^1} = \exp\left(-\frac{2\pi i q^2}{k}\text{Link}(\gamma^0,\gamma^1)\right)W_{\gamma^1}W_{\gamma^0}, \tag{2.50}$$

showing that the phase acquired is exactly the Aharonov-Bohm phase already obtained. It is also possible to write the non-commutativity of the Wilson lines as

$$\begin{aligned}\langle W_{\gamma^0}W_{\gamma^1}\rangle &= \exp\left(\frac{2\pi i q^2}{\kappa}\text{Link}(\gamma^0,\gamma^1)\right)\\ &= e^{\frac{2\pi i}{\kappa}q^2}\langle W_{\gamma^0}\rangle\langle W_{\gamma^1}\rangle,\end{aligned} \tag{2.51}$$

considering that the quasiparticle makes a 2π turn around quasiparticle 1, that is, $\text{Link}(\gamma^0,\gamma^1) = 1$. In other words,

$$\langle W_{\gamma^0}W_{\gamma^1}\rangle_{\text{linked}} = e^{\frac{2\pi i}{\kappa}q^2}\langle W_{\gamma^0}W_{\gamma^1}\rangle_{\text{unlinked}}. \tag{2.52}$$

The expression above shows that when the two curves are linked once (that is, when the Wilson line W_{γ^1} encircles the quasiparticles that is at rest in space), the wavefunction associated with the first particle acquires a phase $e^{\frac{2\pi i}{\kappa}q^2}$, which is the Anaronov-Bohm phase obatined previously. The anyonic phase associated with the exchange of two identical particles is given by half of the Aharonov-Bohm phase, that is $e^{\frac{\pi i}{\kappa}q^2}$. This means that the statistics is given by $\theta = q^2/\kappa$; the same result obtained above.

Therefore, knowing the relation between the statistics and the phase acquired by exchanging two quasiparticles, it is possible to further study the holonomy of the Wilson lines W_{γ^0} and W_{γ^1} using equation 2.50. First, let us summarize our previous results as

$$W_{\gamma^0} W_{\gamma^1} = \exp\left(-\frac{2\pi i q^2}{k}\text{Link}(\gamma^0, \gamma^1)\right) W_{\gamma^1} W_{\gamma^0} \tag{2.53}$$

$$\theta_q = \frac{q^2}{\kappa}\text{mod } 2 \tag{2.54}$$

$$s_q = \frac{\theta_q}{2} = \frac{q^2}{2\kappa}\text{mod } 1, \tag{2.55}$$

where s_q is the spin related to the particle with a_μ-charge q, where the relation that $s = \theta/2$ was used due to the spin-statistics theorem.

Consider the statistics and spin for the curves with $q = n$ and $q = n + k$ as

$$q = n \implies \theta_n = \frac{n^2}{\kappa}\text{mod } 2; \qquad s_n = \frac{n^2}{2\kappa}\text{mod } 1 \tag{2.56}$$

$$q = n + \kappa \implies \theta_{n+\kappa} = \frac{n^2}{\kappa} + \kappa \text{ mod } 2; \qquad s_{n+\kappa} = \frac{n^2}{2\kappa} + \frac{\kappa}{2}\text{mod } 1. \tag{2.57}$$

Due to the statistics being defined modulo 2, if κ is even, the additional contribution to the statistics from the curve $n+\kappa$ can be absorbed into the mod 2 structure. Also, in the case κ is even, the spins associated with the curves n and $n + \kappa$ are the same because of the modulo function. Therefore, when κ is even, the curves n and $n + \kappa$ yield the same holonomy and are equivalent. Thus, the theory has κ independent lines.

In the case where $\kappa = 2l + 1$ is odd with $l = 0, 1, \ldots,$ notice that

$$\theta_{n+\kappa} = \frac{n^2}{\kappa} + \kappa \text{ mod } 2; \qquad s_{n+\kappa} = \frac{n^2}{2\kappa} + \frac{\kappa}{2}\text{mod } 1,$$

and substituting $\kappa = 2l + 1$

$$\theta_{n+\kappa} = \frac{n^2}{\kappa} + 1 \text{ mod } 2; \qquad s_{n+\kappa} = \frac{n^2}{2\kappa} + \frac{1}{2}\text{mod } 1.$$

Through this expression, we can see that the statistics of the curve with $q = n + \kappa$ differ from the statistics of the curve n by 1mod 2, and therefore they represent different quasiparticles. In addition, the spin of the curve $n + \kappa$ differs from the spin of n by 1/2 mod 1. Thus, despite the curves n and $n + \kappa$ yielding the same holonomy for κ odd, it is not possible to say that these curves are equivalent.

Notice that when κ is odd the curve $n + 2\kappa$ have the following spin and statistics

$$\theta_{n+2\kappa} = \frac{(n + 2\kappa)^2}{\kappa}\text{mod } 2 = \frac{n^2}{\kappa}\text{mod } 2$$

$$s_{n+2\kappa} = \frac{(n + 2\kappa)^2}{2\kappa}\text{mod } 1 = \frac{n^2}{2\kappa}\text{mod } 1,$$

and, in this case, the statistics and the spin are the same for the curve n. Thus, these lines are equivalent, and the system has 2κ independent lines when κ is odd.

In particular, a line with $q = \kappa = 2l + 1$ odd has statistics

$$\begin{aligned}\theta_\kappa &= \frac{\kappa^2}{\kappa}\text{mod } 2 = \kappa \text{ mod } 2 = 1 + 2l \text{ mod } 2, \\ &= 1 \text{ mod } 2\end{aligned} \tag{2.58}$$

and spin

$$\begin{aligned}s_\kappa &= \frac{\kappa^2}{2\kappa}\text{mod } 1 = \frac{\kappa}{2} \text{ mod } 2 = \frac{1}{2} + l \text{ mod } 1 \\ &= \frac{1}{2} \text{ mod } 1.\end{aligned} \tag{2.59}$$

Thus, the line $2l + 1$ has fermionic statistics and spin, which means that this Wilson line describes the world line of a fermion.

2.6 Chern-Simons in the Torus

In this section, we aim to investigate the ground-state degeneracy of a Chern-Simons theory defined on a torus. For a Chern-Simons theory defined in a plane $\mathbb{R} \times \mathbb{R}^2$, the ground-state is unique. However, the torus is a two-dimensional surface with a non-trivial topology, and therefore the ground-state will be degenerate and dependent on the topology. This property is called a topological degeneracy.

Consider a torus as indicated in Figure 5, where the non-contractible loops C_1 and C_2 are identified as opposite edges of a square. The first loop C_1 winds around the square in the x_1-direction, and C_2 winds around in the x_2-direction. Consider two unitary operators T_1 and T_2 with the following properties:

- W_1 describes the process of creating a pair of particles-antiparticles, moving each one in opposite directions around the C_1 loop until they annihilate each other on the opposite side of the torus. This process is indicated in Figure 6.
- W_2 describes the process of creating a pair of particles-antiparticles, moving each one in opposite directions around the C_2 loop until they annihilate each other on the opposite side of the torus.

The inverse operators W_1^{-1} and W_2^{-1} describe time-reversed processes. In addition, both W_1 and W_2 preserve the ground-state of the system, since these processes begin and end in the vacuum. Therefore, no net excitation is introduced into the system.

Consider the process

$$W_2^{-1}W_1^{-1}W_2W_1,$$

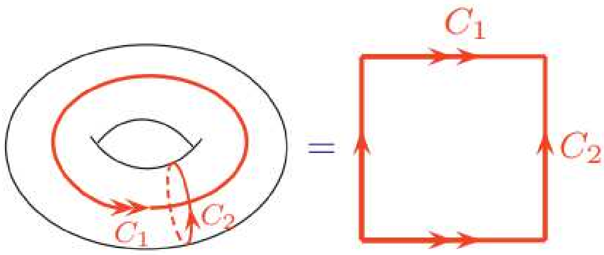


Figura 5 – A torus with the two non-contractible loops indicated as C_1 and C_2. Both non-contractible cycles can be identified as the edges of a square, as indicated in the right drawing. From Reference [3].

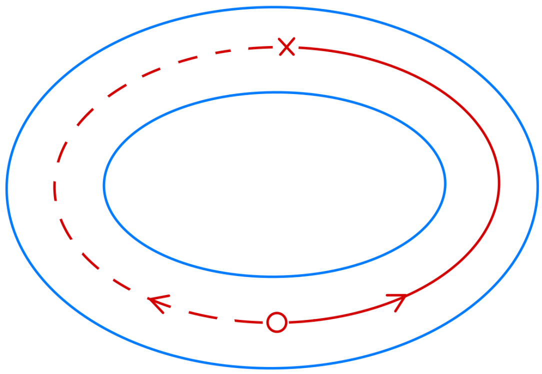


Figura 6 – Illustration of the process W_1 of creating a particle-antiparticle pair is represented by the red dot in the figure, and their annihilation is indicated at the point marked 'x'. This process is happening in the loop C_1 of the torus.

where the time is read from the right to the left. The process is performed in the following steps:

1. W_1 perform a loop along C_1;
2. W_2 perform a loop around C_2;
3. W_1 perform the inverse process indicated in (1);
4. W_2 perform the inverse process indicated in (2),

where the operators W_1 and W_2 can be understood as Wilson operators along the directions C_1 and C_2, respectively. This process is represented in Figure 7.

The process $W_2^{-1}W_1^{-1}W_2W_1$ induces a phase $e^{2i\theta}$. However, notice that if we commute the intermediary process $W^{-1}W_2$, we get the following process

$$W_2^{-1}W_2W_1^{-1}W_1 = \mathbb{I}$$

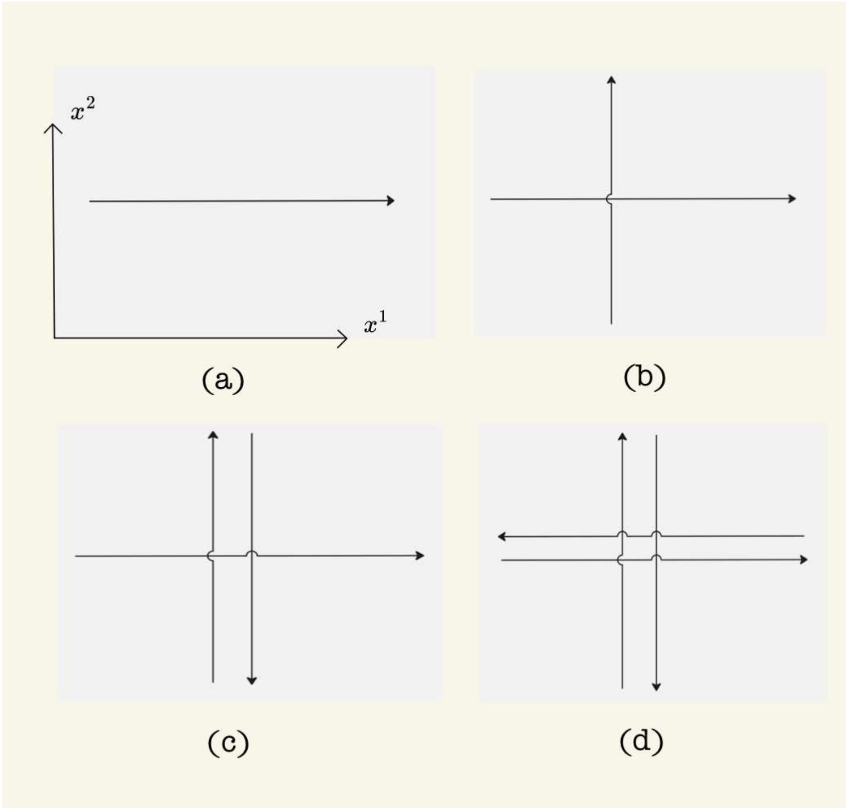


Figura 7 – Process $W_2^{-1}W_1^{-1}W_2W_1$ in steps, where (a). a quasiparticle world line sweeps through C_1; (b). a quasiparticle world line sweeps through C_2; (c). a quasiparticle traverses C_1 in the reverse order; (d). a quasiparticle traverses C_2 in the reverse order.

and the curves are unlinked. Therefore, to find the phase in the linked process, we must calculate

$$W_1^{-1}W_2 = e^{-i\int a_1}e^{i\int a_2} = e^{-[\int a_1, \int a_2]}W_2W_1^{-1}. \tag{2.60}$$

Using equation 2.43, one has

$$W_1^{-1}W_2 = e^{2\pi i/\kappa}W_2W_1^{-1}, \tag{2.61}$$

and substituting this in the process $W_2^{-1}W_1^{-1}W_2W_1$, one gets

$$W_2^{-1}W_1^{-1}W_2W_1 = e^{2\pi i/\kappa}W_2^{-1}W_2W_1^{-1}W_1 = e^{2\pi i/\kappa}. \tag{2.62}$$

Although the Wilson operators W_1 and W_2 in general do not commute[3], both of these operators commute with the Hamiltonian. Therefore, they are symmetries of the system, and, as a result, the theory has a ground-state degeneracy in the torus.

Let us show how many degenerate ground-states a Chern-Simons with level κ has in the torus. As W_1 is a unitary operator, its eigenvalue is a phase $e^{i\alpha}$ where $\alpha \in [0, 2\pi)$.

[3] The cases where W_1 and W_2 commute are when $\theta = 0$, representing the world line of bosons, and $\theta = \pi$, which represents the world line of fermions.

The action of this operator is

$$W_1 |\alpha\rangle = e^{i\alpha} |\alpha\rangle . \tag{2.63}$$

Since W_2 also commutes with the Hamiltonian, the state $W_2 |\alpha\rangle$ is also in the ground-state space of our system.

Then, apply W_1 to this new state using

$$W_1 W_2 = e^{-2\pi i/\kappa} W_2 W_1, \tag{2.64}$$

as

$$\begin{aligned} W_1(W_2 |\alpha\rangle) &= e^{-2\pi i/\kappa} W_2 W_1 |\alpha\rangle = e^{-2\pi i/\kappa} e^{i\alpha} W_2 |\alpha\rangle \\ &= e^{i\alpha - 2\pi i/\kappa} W_2 |\alpha\rangle . \end{aligned} \tag{2.65}$$

Renaming $W_2 |\alpha\rangle \to |\alpha - 2\pi/\kappa\rangle$, one has

$$W_1 |\alpha - 2\pi/\kappa\rangle = e^{i\alpha - 2\pi i/\kappa} |\alpha - 2\pi/\kappa\rangle , \tag{2.66}$$

and we generated another ground-state. We can continue to generate states by iteration as

$$|\alpha\rangle , |\alpha - 2\pi/\kappa\rangle , |\alpha - 4\pi/\kappa\rangle , \dots , |\alpha - 2(\kappa - 1)\pi/\kappa\rangle . \tag{2.67}$$

The state corresponding to $|\alpha - 2\pi\rangle$, which would be the next state in the sequence in 2.67, is equivalent to $|\alpha\rangle$ as they differ by $e^{-2\pi i} = 1$. Thus, there are a number κ of degenerate ground-states.

2.7 Hierarchical States

The theory developed so far concerns the first level of an FQH system. It is possible to construct a second-level hierarchical FQH state that contains two kinds of quasiparticles. To do that, instead of having one emergent dynamical field a_μ, we will have an additional emergent dynamical gauge field $\bar{a}_\mu$ coupled to a matter current $\bar{j}^\mu$ with

$$\bar{j}^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} \partial_\nu \bar{a}_\rho . \tag{2.68}$$

The effective action coupled to a background electromagnetic field A_μ is given by

$$\begin{aligned} S[a, \bar{a}, A] &= \int_{\mathcal{M}} \frac{\kappa_1}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{\kappa_2}{4\pi} \varepsilon^{\mu\nu\rho} \bar{a}_\mu \partial_\nu \bar{a}_\rho - A_\mu J^\mu - a_\mu \bar{j}^\mu \\ &= \int_{\mathcal{M}} \frac{\kappa_1}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{\kappa_2}{4\pi} \varepsilon^{\mu\nu\rho} \bar{a}_\mu \partial_\nu \bar{a}_\rho - \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho - \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu \bar{a}_\rho , \end{aligned} \tag{2.69}$$

and notice that the gauge field $\bar{a}_\mu$ is dynamical, meaning that it has a CS term itself with level κ_2. The topological current that couples to the electromagnetic field is given by

$$J^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho , \tag{2.70}$$

which is conserved.

The Euler-Lagrange equations for a_0 and $\bar{a}_0$ are given by

$$\partial_\nu \frac{\partial \mathcal{L}_{CS}}{\partial(\partial_\nu\, a_0)} - \frac{\partial \mathcal{L}_{CS}}{\partial a_0} = 0 \tag{2.71}$$

$$\partial_\nu \frac{\partial \mathcal{L}_{CS}}{\partial(\partial_\nu\, \bar{a}_0)} - \frac{\partial \mathcal{L}_{CS}}{\partial \bar{a}_0} = 0. \tag{2.72}$$

The equation of motion for a_0 is obtained by using equation 2.71 and the Lagrangian in equation 2.69 as

$$\frac{\kappa_1}{4\pi}\varepsilon^{\mu\nu 0}\partial_\nu a_\mu - \frac{\kappa_1}{4\pi}\varepsilon^{0\nu\rho}\partial_\nu a_\rho - \frac{1}{2\pi}\varepsilon^{\mu\nu 0}\partial_\nu A_\mu + \frac{1}{2\pi}\varepsilon^{0\nu\rho}\partial_\nu \bar{a}_\rho = 0 \tag{2.73}$$

$$-\frac{\kappa_1}{2\pi}\varepsilon^{0ij}\partial_i a_j + \frac{1}{2\pi}\varepsilon^{ij}\partial_i A_j + \frac{1}{2\pi}\varepsilon^{0ij}\partial_i \bar{a}_j = 0 \tag{2.74}$$

$$\partial_i a_j = \frac{1}{\kappa_1}\partial_i A_j + \frac{1}{\kappa_1}\partial_i \bar{a}_j. \tag{2.75}$$

Defining the magnetic fields associated with each gauge field as

$$b = \varepsilon^{ij}\partial_i a_j \tag{2.76}$$

$$\bar{b} = \varepsilon^{ij}\partial_i \bar{a}_j \tag{2.77}$$

$$B = \varepsilon^{ij}\partial_i A_j, \tag{2.78}$$

the EoM for a_μ in equation 2.74 becomes

$$-\kappa_1 b + B + \bar{b} = 0$$

$$B = \kappa_1 b - \bar{b}. \tag{2.79}$$

We can also find the EoM for the gauge field $\bar{a}$ by following the same procedure for $\bar{a}_0$ through equation 2.72 as

$$-\frac{\kappa_2}{2\pi}\varepsilon^{ij}\partial_i \bar{a}_j + \frac{1}{2\pi}\varepsilon^{ij}\partial_i a_j = 0$$

$$\partial_i \bar{a}_j = \frac{1}{\kappa_2}\partial_i a_j \tag{2.80}$$

$$\kappa_2 \bar{b} = b. \tag{2.81}$$

We aim to extract topological properties of the hierarchical CS system, such as the Hall conductivity and the filling fraction. To do that, consider the temporal component of the electromagnetic current as

$$j^0 = \rho = -\frac{\delta \mathcal{L}}{\delta A_0} = \frac{1}{2\pi}\varepsilon^{ij}\partial_i a_j = \frac{b}{2\pi}, \tag{2.82}$$

which is related to the attribute of flux to the charges of the system. The Hall conductivity can be found from the charge density ρ as

$$\sigma_{xy} = \frac{\rho}{B} = \frac{b}{2\pi B},$$

and using equations 2.79 and 2.81 to simplify this expression, the Hall conductivity is given by

$$\begin{aligned}\sigma_{xy} &= \frac{b}{2\pi(\kappa_1 b - \bar{b})} = \frac{\kappa_2 \bar{b}}{2\pi\bar{b}(\kappa_1\kappa_2 - 1)} \\ &= \frac{1}{2\pi\left(\kappa_1 - \frac{1}{\kappa_2}\right)}.\end{aligned} \tag{2.83}$$

As a result of the form of the Hall conductivity in equation 2.83, the theory represented in equation 2.69 describes an effective theory of a second-level hierarchical state. It is also possible to obtain the same result for the Hall conductivity using a more laborious approach by integrating the field $\bar{a}_\mu$ out of the action, followed by integrating out the field a_μ to obtain a CS term for the electromagnetic field A_μ.

The generalization for higher hierarchies is accomplished similarly by introducing new emergent gauge fields with a Chern-Simons term. For this case, the filling factor is given by

$$\nu = \frac{1}{\kappa_1 \pm \frac{1}{\kappa_2 \pm \frac{1}{\kappa_3 \pm \dots}}} \tag{2.84}$$

The Chern-Simons action of these theories can be written in a more compact way using the K-matrix formulation. For example, the second-level CS theory can be described by the compact action

$$S[a] = \int d^3x \frac{1}{4\pi} K_{IJ} \varepsilon^{\mu\nu\rho} a^I_\mu \partial_\nu a^J_\rho - \frac{1}{2\pi} t_I \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu a^I_\rho, \tag{2.85}$$

with $I, J = 1, 2$, K-matrix

$$K = \begin{bmatrix} \kappa_1 & -1 \\ -1 & \kappa_2 \end{bmatrix}, \tag{2.86}$$

and the charge vector

$$t = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.87}$$

By using this notation, it is possible to realize that this action is indeed the compact form of equation 2.69, with $(a^1_\mu, a^2_\mu) = (a_\mu, \bar{a}_\mu)$. The K-matrix formulation, in fact, is used to describe *any* Abelian anyon theory, not just the hierarchical states. This formulation will be covered in more detail in Chapter 4.

3 $U(1)$ EDGE CHERN-SIMONS THEORY

In this chapter, we develop the single-component $U(1)$ Chern-Simons edge theory. We will show that when the bulk, which is defined in a manifold $\mathcal{M}$, has a boundary $\partial\mathcal{M}$, the theory will have gapless edge mode excitations at the edge, as a result of a gauge anomaly. The main properties of this system will also be presented.

3.1 Edge Lagrangian

It was already seen that the bulk CS action in 2+1 dimensions is defined in a manifold $\mathcal{M}$, and it is given by

$$S_{\text{bulk}} = \int_{\mathcal{M}} d^3x \frac{\kappa}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho. \tag{3.1}$$

This bulk action is gauge-invariant up to boundary terms, which is a gauge anomaly of the theory. Therefore, S_{bulk} is strictly invariant only if the manifold $\mathcal{M}$ has no boundary or if the boundary terms can be neglected. In the case where $\mathcal{M}$ has a boundary $\partial\mathcal{M}$ and the boundary terms cannot be neglected, then it is expected that there will be physical (dynamical) modes at the edge of the theory. Thus, the dynamical edge excitations are required to cancel the gauge anomaly in the bulk theory.

We will derive the CS edge theory using bosonization in 1+1 dimensions. For that, consider the manifold $\mathcal{M}$ decomposed as

$$\mathcal{M} = \mathbb{R} \times \Sigma,$$

with

$$\Sigma = (-\infty, 0] \times \mathbb{R},$$

where $x^1 \in (-\infty, 0]$, so that there is a physical boundary at $x^1 = 0$, as shown in Figure 8, where the bulk is represented in the bottom semi-plane. Also, assume the pure-gauge configuration[1] in the bulk as

$$a_\mu = \partial_\mu \phi, \tag{3.2}$$

with compactness of the scalar field, that is, $\phi \sim \phi + 2\pi n$; $n \in \mathbb{Z}$.

Consider the commutation relation already derived in the last chapter as

$$[a_i(\vec{x}), a_j(\vec{x}')] = \frac{2\pi i}{\kappa} \varepsilon_{ij} \delta^2(\vec{x} - \vec{x}'), \tag{3.3}$$

and substitute the pure gauge configuration into equation 3.2 to obtain

$$[\partial_i \phi(\vec{x}), \partial_j \phi(\vec{y})] = \frac{2\pi i}{\kappa} \varepsilon_{ij} \delta(x^1 - y^1)\delta(x^2 - y^2). \tag{3.4}$$

[1] The pure-gauge choice was made through the equation of motion $da = f$ in which the solutions are the pure-gauge configuration.

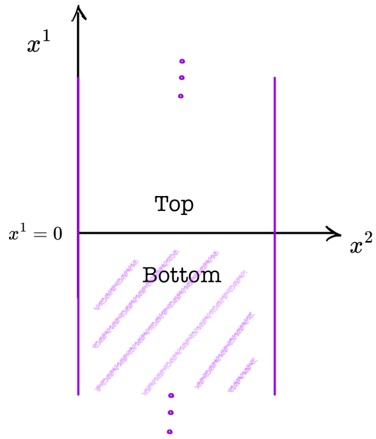


Figura 8 – Illustration of the bottom edge of a Chern-Simons theory. In the figure, the bulk corresponds to the x_1-x_2 plane where $x_1 \in (-\infty, 0]$ (represented in the pink-shaded region) while the edge is located at $x_1 = 0$.

In the next step, we integrate the commutator in equation 3.4 over x^1 in the interval $x^1 \in (-\infty, 0]$ to get

$$[\phi(x^2), \partial_{y^2}\phi(y^2)] = \frac{2\pi i}{\kappa}\delta(x^2 - y^2). \tag{3.5}$$

Assuming $x^2 = x$ and $y^2 = y$ to simplify, we have

$$[\phi(x), \partial_y\phi(y)] = \frac{2\pi i}{\kappa}\delta(x - y), \tag{3.6}$$

which is the Kac-Moody algebra for the edge fields ϕ. This expression is the commutator relation for a chiral boson field. Therefore, ϕ is a one-way propagating boson field. This result can be seen more explicitly through the equation of motion, which will be presented below.

Let us write the commutator in terms of the fields only. To do that, integrate this commutation relation over y to get[2]

$$[\phi(x), \phi(y)] = \frac{\pi i}{\kappa}\text{sign}(x - y), \tag{3.7}$$

and suppose that $\partial_y\phi(y)$ is a function of $\Pi(y)$, that is,

$$\partial_y\phi(y) = a\Pi(y), \tag{3.8}$$

where a is a constant coefficient to be determined. Applying this relation to the commutator in equation 3.6, one has

$$a = \frac{4\pi}{\kappa}, \tag{3.9}$$

[2] The integration in y of $\delta(x - y)$ gives a step function, which in this case leads to the sign function, since x and y are symmetric in the commutator.

and therefore

$$\Pi(x) = \frac{\kappa}{4\pi}\partial_x\phi(x). \tag{3.10}$$

As the conjugate momentum relates to the Lagrangian as

$$\Pi = \frac{\partial L}{\partial(\partial_t\phi)}, \tag{3.11}$$

then the Lagrangian term that gives rise to this momentum is given by

$$L^0_{\text{edge}} = \frac{\kappa}{4\pi}\partial_t\phi\partial_x\phi. \tag{3.12}$$

However, note that this Lagrangian has no propagating degrees of freedom, as it leads to a null Hamiltonian. Since we are assuming that there are dynamical degrees of freedom in the edge, we must add a propagating term compatible with the symmetries of the system. An adequate choice would be a quadratic term in momenta, that is,

$$\begin{aligned} L_{\text{edge}} &= L^0_{\text{edge}} + L^1_{\text{edge}} \\ &= \frac{\kappa}{4\pi}\partial_t\phi\partial_x\phi - \frac{\kappa v}{4\pi}\partial_x\phi\partial_x\phi \\ &= \frac{\kappa}{4\pi}\left(\partial_t\phi\partial_x\phi - v(\partial_x\phi)^2\right), \end{aligned} \tag{3.13}$$

where v is a constant related to the value of the velocity of the edge mode ϕ.

The corresponding Hamiltonian density

$$H = \int dx \frac{\kappa v}{4\pi}(\partial_x\phi)^2 \tag{3.14}$$

must be bounded from below, i.e., it must also be positive-definite. For that to happen, κv must be strictly positive. Therefore, the sign of the level κ of the theory determines the sign of the velocity constant v of the edge modes. For $\kappa > 0$, the velocity v must be strictly positive and the edge modes propagate to the left at the boundary.

The equation of motion associated with ϕ is obtained through

$$\partial_\beta \frac{\partial\mathcal{L}_{CS}}{\partial(\partial_\beta\,\phi)} - \frac{\partial\mathcal{L}_{CS}}{\partial\phi} = 0, \tag{3.15}$$

as

$$(\partial_t - v\partial_x)\partial_x\phi = 0, \tag{3.16}$$

which is the EoM for a chiral boson theory where the excitations are gapless. A very special property of a chiral edge is its robustness against backscattering perturbations. Since the fields are chiral, assume that all modes on the top edge move to the left. Then, if we wanted to apply a backscattering[3] perturbation term to scatter these modes to right-moving modes, these excitations would have to cross the bulk to arrive at the bottom edge.

[3] Backscattering is defined as a physical phenomenon that reflects/scatters particles back from the direction of propagation that they originally came from.

This robustness can be broken if we consider a system with two edges (top and bottom) and the bulk theory between them, as illustrated in Figure 9. When the bulk is extensive enough, this type of scattering would be highly suppressed, and both edges would be robust against these perturbations. However, if the bulk is thin enough, the counter-propagating modes at each edge can interact with one another, and backscattering can occur through a tunneling effect.

Figura 9 – Illustration for a Quantum Hall system with a bulk theory and two edges, top and bottom.

3.2 Vertex Operators

The bulk-edge correspondence associates a quasiparticle in the bulk with an operator at the edge, which is a vertex operator. These vertex operators are related to Wilson lines at the edge, as we will see below.

Consider a Wilson line along an open line in the x^2-direction coming from minus infinity to the edge is given by

$$\begin{aligned} W_n(x^0, x^2) &= \exp\left(in \int_{-\infty}^{0} dx^1 a_1\right) \\ &= \exp\left(in \int_{-\infty}^{0} dx^1 \partial_1 \phi(x^0, x^1, x^2)\right) \\ &= \exp\Big(in\phi(x^0, x^2)\Big), \end{aligned} \tag{3.17}$$

where we have assumed that $\phi \to 0$ when $x^1 \to -\infty$. This operator is known as the vertex operator in Conformal Field Theory, or as the electron operator

$$\Psi_n(x^0, x^2) =: e^{in\phi} :, \tag{3.18}$$

which is related to the excitations at the edge. This operator has normal ordering, which will be omitted from now on for notational simplicity. Each particle n in the bulk has a corresponding vertex operator Ψ_n at the edge.

With the commutator of the field ϕ, we can calculate the algebra of the vertex operators as

$$\Psi_n \Psi_m = \exp\left(-\frac{i\pi}{\kappa} nm \ \mathrm{sign}(x^2 - y^2)\right) \Psi_m \Psi_n, \tag{3.19}$$

with $\Psi_n = e^{in\phi(x^2)}$ and $\Psi_m = e^{im\phi(y^2)}$. Therefore, the exchange phase of a vertex operator Ψ_n is given by $\exp(-i\pi n^2/\kappa)$ and the statistics is

$$\theta_n = \frac{n^2}{\kappa}. \tag{3.20}$$

Notice that when $n = \kappa$ is odd, the statistics of the vertex operator is fermionic. Thus, this vertex operator is, in fact, a fermion. In the simplest case when $\kappa = 1$, it is possible to fermionize the theory using

$$\psi = \frac{1}{\sqrt{2\pi a}}\Psi_{\kappa=1}, \tag{3.21}$$

with a being the short-distance cut-off.

The richness of the edge excitations corresponds to the topologically rich properties of the bulk theory. This is due to the bulk-edge correspondence. For example, distinct topological orders in the bulk lead to different structures of the edge excitations.

3.3 Coupling to Electromagnetic Field

In this section, we aim to analyze the problem with gauge-invariance that arise when we consider the usual minimal coupling with the electromagnetic field A_μ and how to get around this problem by changing the form of the coupling.

It was already seen that the theory is coupled to an electromagnetic field A_μ through

$$\begin{aligned} S_{\text{coupling}} &= \int_M d^3x A_\mu J^\mu \\ &= \frac{1}{2\pi}\int_M d^3x \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho. \end{aligned} \tag{3.22}$$

This expression is not gauge-invariant under a gauge transformation of the field A_μ in the presence of a boundary, despite being invariant under gauge transformations of the emergent field a_μ.

We can redefine the coupling by performing an integration by parts in equation 3.22 and throwing away the boundary term that breaks gauge invariance. Then, one has the new gauge-invariant coupling as

$$S_{\text{coupling}} = -\frac{1}{2\pi}\int_M d^3x\, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho. \tag{3.23}$$

This new coupling is invariant under gauge transformations of A_μ and under restricted[4] gauge transformations of a_μ. Therefore, in the presence of a boundary, this is the correct coupling of an electromagnetic field [18].

---

[4] The gauge transformation $a_\mu \to a_\mu + \partial\lambda$ is restricted and we impose that the transformation parameter vanishes at the boundary, that is, $\lambda|\partial M = 0$).

Recall that in the previous results, we were considering $x^2 = x$ for simplicity. We return to the usual notation x^0, x^1, and x^2 to avoid confusion in the next steps. Set $A_1 = 0$ and consider the background fields A_0 and A_2 that are independent of the variable x^1. Also, fix the gauge $a_0 = 0$. Then

$$\begin{aligned} S_{\text{coupling}} &= -\frac{1}{2\pi}\int_M d^3x\, \varepsilon^{1\nu\rho} a_1 \partial_\nu A_\rho \\ &= \frac{1}{2\pi}\int_M d^3x\, a_1(\partial_0 A_2 - \partial_2 A_0) \\ &= \frac{1}{2\pi}\int_M d^3x\, \partial_1\phi(\partial_0 A_2 - \partial_2 A_0), \end{aligned} \tag{3.24}$$

and integrating over x^1 followed by an integration by parts, one has

$$\begin{aligned} S_{\text{coupling}} &= \frac{1}{2\pi}\int_{\partial M} dx^0 dx^2\, \phi(\partial_0 A_2 - \partial_2 A_0) \\ &= \frac{1}{2\pi}\Big(-\int_{\partial M} d^2x\, A_2\partial_0\phi - \int_{\partial M} d^2x\, \phi\partial_2 A_0)\Big) \\ &= \frac{1}{2\pi}\Big(-\int_{\partial M} d^2x\, vA_2\partial_2\phi + \int_{\partial M} d^2x\, A_0\partial_2\phi)\Big) \\ &= \frac{1}{2\pi}\int_{\partial M} d^2x\, (A_0 - vA_2)\partial_2\phi, \end{aligned} \tag{3.25}$$

where we used the equation of motion to fix the constraint $\partial_0\phi = v\partial_2\phi$ on the first term.

The expression in equation 3.25 indicates that the charge density ρ at the edge $x^1 = 0$ that couples to the temporal component of the electromagnetic field A_0 is given by

$$\rho = \frac{1}{2\pi}\partial_2\phi = \frac{1}{2\pi}\frac{\partial\phi}{\partial x^2}. \tag{3.26}$$

By using the simplification $x^2 = x$, we have

$$\rho = \frac{1}{2\pi}\partial_x\phi, \tag{3.27}$$

which can also be written as

$$\rho(x) = \frac{1}{2\pi} : \Psi^\dagger(x)\Psi(x) :, \tag{3.28}$$

where $: \, :$ indicates normal ordering, which will be omitted from now on. On the other hand, the current density is the term that couples with the spatial component of A and is given by $j = -v\rho$, which is a chiral current flowing through the edge.

It is possible to calculate the quasiparticle charge through the commutation relation for the fields ϕ in equation 3.6 by using the relation for ρ in equation 3.27 as

$$[\rho(x), \Psi_n^\dagger(y)] = \frac{n}{\kappa}\delta(x-y)\Psi_n^\dagger(y) \tag{3.29}$$

with Ψ_n given by equation 3.18. From this expression, it follows that the quasiparticle $\Psi_n^\dagger$ carries charge n/κ, which is generally fractional.

In the case $n = \kappa$, from equation 3.30, we infer that the operator $\Psi_\kappa^\dagger$ has unit charge and corresponds to creating the underlying particle (boson for κ even and fermion for κ odd) at the edge with

$$[\rho(x), \Psi_n^\dagger(y)] = \delta(x-y)\Psi_n^\dagger(y). \tag{3.30}$$

Therefore, the vertex operator Ψ_κ is related to removing particles from the edge. This can be seen more explicitly by integrating equation 3.30 over x to obtain

$$[\hat{N}, \Psi^\dagger(y)] = \Psi^\dagger(y),$$

which means that

$$\hat{N}\Psi^\dagger(y)\left|N\right\rangle = (N+1)\Psi^\dagger(y)\left|N\right\rangle,$$

where $\hat{N} = \int dx \rho(x)$ is the particle number operator.

In the particular case $n = 1$, the vertex operator is

$$\Psi = e^{i\phi},$$

which is the simplest vertex operator in the theory. The creation operator $\Psi^\dagger$ has a commutation relation to the charge density given as

$$[\rho(x), \Psi^\dagger(y)] = \frac{1}{\kappa}\delta(x-y)\Psi^\dagger(y), \tag{3.31}$$

and corresponds to the creation of charges $1/\kappa$, as in a quantum Hall fluid. These quasiparticles with fractional charge have an exchange phase $e^{-i\pi/\kappa}$, corresponding to the statistics of anyons.

In conclusion, the edge must have non-trivial dynamical degrees of freedom to restore gauge invariance. As a result, a current density flows through the edge. In terms of charge density, the equation of motion becomes

$$\partial_0\rho - v\partial_1\rho = 0, \tag{3.32}$$

which is a wave-type equation for ρ. This means that a chiral wave propagates at speed v at the edge. Notice that this wave equation has solutions of the form $\rho(x+vt)$, but counter-propagating waves of type $\rho(x-vt)$ are not allowed solutions. Because of that, the same result previously obtained that perturbation terms, as backscattering, are not allowed, and the edge is robust against perturbation is obtained again by analyzing the edge currents. In other words, the equation of motion in equation 3.32 holds even in the presence of interactions, assuring that the edge modes will remain gapless.

Classically, the rise of a current density at the edge can be understood by looking at the movement performed by the quasiparticles in a quantum Hall liquid in a spatial plane as represented in Figure 10, where the quasiparticles in the bulk have a cyclotron

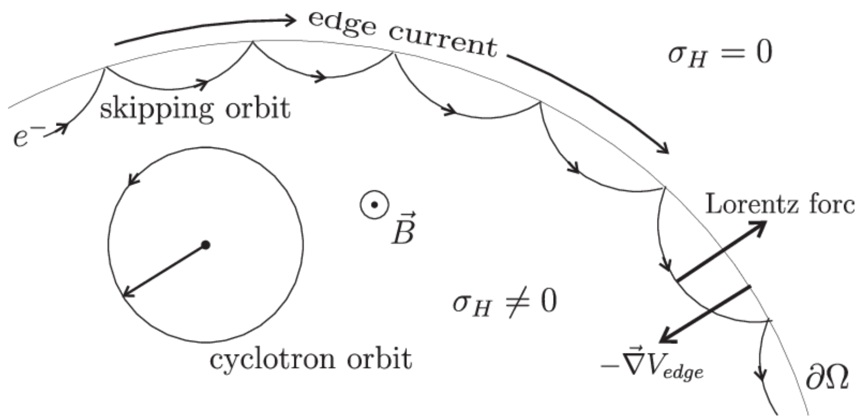


Figura 10 – Illustration of the skipping orbit of the particles near the edge, which generates an edge current. In the bulk, the particles describe a cyclotron orbit. From Reference [4].

orbit, but the quasiparticles near the edge have skipping orbits, which gives rise to an edge current.

Based on this, we conclude that the dynamical boson field living at the edge of a $U(1)$ CS theory has a one-way propagation and the edge is robust against perturbation. This property is related to conformal theory data, which is the central charge of the theory. In the case for the edge theory regarding the bulk at filling $\nu = 1/\kappa$, the central charge is $c = 1$, showing the chirality of the edge mode.

3.4 Thermal Conductivity and Central Charge

In this section, we will analyze how the thermal Hall conductivity at the edge relates to the central charge of the system.

For a pure chiral system in $1+1$ dimensions, the theory is gapless. Since the edge modes are gapless, they can carry energy, leading to thermal transport at the boundary. The chiral edge theory can be described by a conformal field theory. The central charge c is a quantity related to the number of left and right propagating modes at the edge, and, thus, it is an observable of the theory. In the case of a $U(1)$ Chern-Simons theory with level κ, the edge theory is chiral with $c = 1$ (one-way propagation). The propagation direction of the edge modes (left or right) is defined by the sign of the level κ.

The central charge is a well-known concept in conformal theories and is related to the heat capacity per unit length c_v of the related system by

$$c_v = \frac{\pi k_b^2 c T}{6v}, \tag{3.33}$$

where v is the velocity of the edge mode, T is the temperature, k_B is Boltzmann's constant, c is the central charge of the system, and we are assuming $\hbar = 1$.

The energy density ϵ per unit length is the integral of the specific capacity as

$$\epsilon = \int_0^T c_v(T')dT' = \frac{\pi k_b^2 c T^2}{12v}, \tag{3.34}$$

and, therefore, the energy flux in the chiral edge is given by

$$J^q = v\epsilon = \frac{\pi k_b^2 c T^2}{12}, \tag{3.35}$$

which is the thermal Hall effect. The variation of this current is given by

$$\delta J^q = \frac{\pi k_b^2 c T}{6}\delta T. \tag{3.36}$$

The equation 3.36 indicates that the boundary of a Topological Quantum Field Theory can have gapless modes carrying heat. The central charge of this chiral edge is related to the heat transport along the edge. Through the Fourier heat equation

$$\delta J^q = \kappa_T \delta T, \tag{3.37}$$

we can obtain [5]

$$\kappa_T = \frac{\pi k_b^2 T}{6} c, \tag{3.38}$$

which is the thermal Hall conductivity κ_T.

Two different Laughlin states at distinct filling fractions $\nu_1 = 1/\kappa_1$ and $\nu_2 = 1/\kappa_2$ have the same thermal conductance, since this observable is independent of the level κ. This is because both theories have the same central charge $c = 1$, which is independent of κ [90].

[5] Careful to not confuse the thermal Hall conductivity κ_T with the level κ of the theory, since the notation is similar.

4 $U(1)^N$ CHERN-SIMONS THEORY

In this chapter, we will develop an effective field theory for any Abelian topologically ordered state through the K-matrix formulation.

4.1 Multi-Component Action

Consider the following multi-component $U(1)^N$ Chern-Simons (CS) action in 2+1 dimensions

$$S[a] = \int d^3x \frac{1}{4\pi} K_{IJ} \varepsilon^{\mu\nu\rho} a^I_\mu \partial_\nu a^J_\rho, \tag{4.1}$$

where we have assumed that there are a number N of $U(1)$ emergent gauge fields a^I_μ with $I, J = 1, 2, \ldots, N$. Thus $K_{IJ} \in \mathbb{Z}^N \times \mathbb{Z}^N$ is an $N \times N$ symmetric matrix with integer entries, which specifies the CS couplings. Notice that K is a symmetric matrix to maintain the CS action invariant under the exchange of I and J, since

$$\begin{aligned} S[a] &= \int d^3x \frac{1}{4\pi} K_{JI} \varepsilon^{\mu\nu\rho} a^J_\mu \partial_\nu a^I_\rho \\ &= -\int d^3x \frac{1}{4\pi} K_{JI} \varepsilon^{\mu\nu\rho} a^I_\rho \partial_\nu a^J_\mu \\ &= \int d^3x \frac{1}{4\pi} K_{IJ} \varepsilon^{\rho\nu\mu} a^I_\rho \partial_\nu a^J_\mu, \end{aligned}$$

where we used integration by parts in steps 1 to 2, the antisymmetric property of the Levi-Civita symbol in steps 2 to 3. This shows that $K_{IJ} = K_{JI}$, and therefore K is a symmetric matrix. As in the single-component case, where the level κ was an integer value because of large gauge transformations, K_{IJ} has integer entries also because of the requirement for invariance of this action under large gauge transformations.

The $U(1)^N$ Chern-Simons theory is topological just like the single-component theory, since its action does not depend on the metric $g_{\mu\nu}$ when placed in a curved space. Therefore, its invariants are also global quantities related to the topology of the space. The CS action is invariant under smooth deformations of the geometry of the space. In addition, the Hamiltonian of this system is zero as the stress tensor vanishes, indicating that the pure Chern-Simons theory lacks local dynamical degrees of freedom.

Notice that if the K-matrix is diagonal, the theory corresponds to N decoupled theories, with each gauge field having a CS term.

4.2 Coupling to Electromagnetic Field

In this section, we will couple the theory with an electromagnetic field to obtain the electromagnetic induced current and the filling fraction of the state, which are associated with the K-matrix.

Minimally couple the theory to an electromagnetic field A_μ through

$$S[a,A] = \int d^3x \frac{1}{4\pi} K_{IJ} \varepsilon^{\mu\nu\rho} a^I_\mu \partial_\nu a^J_\rho - \frac{1}{2\pi} t_I \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu a^I_\rho \tag{4.2}$$

$$= \int d^3x \frac{1}{4\pi} K_{IJ} \varepsilon^{\mu\nu\rho} a^I_\mu \partial_\nu a^J_\rho - A_\mu J^\mu, \tag{4.3}$$

where we have assumed A_μ as an electromagnetic external field (associated with a $U(1)_A$ global symmetry[1]). The charge vector $t \in \mathbb{Z}^N$ describes the coupling between each internal gauge field a^I_μ with the electromagnetic field A_μ by specifying a topological current as

$$J^\mu = \frac{1}{2\pi} t_I \varepsilon^{\mu\nu\rho} \partial_\nu a^I_\rho. \tag{4.4}$$

In other words, each component t_I of the vector charge indicates the electric charge associated with the gauge field a^I_μ.

We want to integrate out the field a_μ to find the induced electromagnetic current in this system. For that, find the equation of motion for the field using the Euler-Lagrange equation

$$\partial_\alpha \frac{\partial L}{\partial(\partial_\alpha a^K_\beta)} - \frac{\partial L}{\partial a^K_\beta} = 0, \tag{4.5}$$

as

$$\varepsilon^{\mu\nu\rho} \partial_\nu a^I_\rho = (K^{-1})^{IJ} t_J \varepsilon^{\mu\nu\rho} \partial_\nu A_\rho$$
$$a^I_\rho = (K^{-1})^{IJ} t_J A_\rho \qquad \text{(locally)} \tag{4.6}$$

By substituting this expression into the source term in the Lagrangian, one has

$$L[A] = -\frac{1}{4\pi} t_I (K^{-1})^{IJ} t_J \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{4.7}$$

which is a CS term for the electromagnetic field A_μ.

Through equation 4.7, it is possible to obtain the electromagnetic current as

$$\langle J^\sigma_{\text{ind}} \rangle = -\frac{\delta L[A]}{\delta A_\sigma}$$
$$= \frac{1}{2\pi} t_I (K^{-1})^{IJ} t_J \varepsilon^{\sigma\nu\rho} \partial_\nu A_\rho, \tag{4.8}$$

with

$$J^y_{\text{ind}} = \frac{1}{2\pi} \sigma^{xy} E_x, \tag{4.9}$$

[1] This global $U(1)$ symmetry associated with the external electromagnetic field is different from the local $U(1)_a$ gauge symmetries (local redundancies) of the internal gauge fields

where σ^{xy} is the Hall conductance and E^x is the electric field in the $x-$direction. Taking the $y-$direction in the electromagnetic current in equation 4.8 and using $F_{0x} = -E_x = \partial_0 A_x - \partial_x A_0$, one gets

$$\langle J^y_{\text{ind}} \rangle = \frac{1}{2\pi} t_I (K^{-1})^{IJ} t_J E^x, \tag{4.10}$$

which is the quantum Hall effect, where the electric field in the x-direction induces an electromagnetic current in the y-direction.

We can compare equation 4.10 to the expression in equation 4.9 to obtain the Hall conductivity as

$$\sigma = t_I (K^{-1})^{IJ} t_J, \tag{4.11}$$

and therefore the expression for the filling fraction of the system is

$$\nu = t^T K^{-1} t. \tag{4.12}$$

4.3 Coupling to Matter Current

In this section, we aim to couple the $U(1)^N$ Chern-Simons theory to a matter current to extract the electromagnetic charge of the quasiparticles $\mathbf{l}$, which in general are fractional, and the charge-flux relation, where each quasiparticle has a flux attached to it.

For that, couple the gauge field a_μ to a quasiparticle current j^μ, considering a background field A_μ, into the Lagrangian as

$$L = \frac{1}{4\pi} K_{IJ} \varepsilon^{\mu\nu\rho} a^I_\mu \partial_\nu a^J_\rho - \frac{1}{2\pi} t_I \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu a^I_\rho - l_I a^I_\mu j^\mu, \tag{4.13}$$

with

$$j^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} \partial_\nu c_\rho, \tag{4.14}$$

where j^μ represents the quasiparticle current, c_ρ is a new single-component auxiliary gauge field. A quasiparticle is represented by a vector $\mathbf{l}$ in the lattice $\Gamma = \mathbb{Z}^N$. Then, the component $l_I \in \mathbb{Z}$ is an integer that represents the internal gauge charge. Therefore, l_I represents the coupling between the quasiparticle and the gauge field a^I_μ. The quasiparticle current represents the worldlines of the gapped quasiparticles [17].

From the Lagrangian with the current term, let us integrate out the field a^I_μ to obtain the charge carried by the quasiparticles. Using the Euler-Lagrange equation for the field a, the equation of motion associated with this field is

$$\frac{1}{2\pi} \varepsilon^{\mu\nu\rho} \partial_\nu a^I_\rho = \frac{(K^{-1})^{IJ}}{2\pi} t_J \varepsilon^{\mu\nu\rho} \partial_\nu A_\rho + (K^{-1})^{IJ} l_J j^\mu. \tag{4.15}$$

Substituting this expression into the Lagrangian term

$$-\frac{1}{2\pi} t_I \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu a^I_\rho, \tag{4.16}$$

we get

$$L[A,c] = -t_I (K^{-1})^{IJ} l_J A_\mu j^\mu + \dots, \tag{4.17}$$

where terms depending on other fields were omitted. This term is responsible for the electric charge carried by a quasiparticle excitation l in response to the electromagnetic field as

$$\begin{aligned} q_l &= t_I (K^{-1})^{IJ} l_J && (4.18)\\ &= t^T K^{-1} l. && (4.19) \end{aligned}$$

The charge-flux relation, analogous to the single-component case, can be obtain from equation 4.13 by turning off the background field $A_\mu = 0$ to obtain

$$L[a] = \frac{1}{4\pi} K_{IJ} \varepsilon^{\mu\nu\rho} a_\mu^I \partial_\nu a_\rho^J - l_I a_\mu^I j^\mu, \tag{4.20}$$

and finding the equation of motion for a by using equation 4.5 as

$$\frac{1}{2\pi} K_{IJ} \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho^I = l_J j^\mu. \tag{4.21}$$

Define the current of the J-th quasiparticle as

$$j_J^\mu = l_J j^\mu.$$

Then, the equation of motion becomes

$$j_J^\mu = \frac{1}{2\pi} K_{IJ} \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho^I, \tag{4.22}$$

which is a conserved topological current.

To elucidate the underlying charge–flux correspondence, it is convenient to rewrite the current explicitly as a function of the magnetic field. For that, define N distinct effective magnetic fields b^I associated with the I-th emergent gauge field as

$$b^I = \varepsilon^{ij} \partial_i a_j^I. \tag{4.23}$$

Using the expression for the magnetic field in the equation of motion, the time-component of the current is given by

$$\begin{aligned} j_J^0 = \rho_J &= \frac{1}{2\pi} K_{IJ} \varepsilon^{ij} \partial_i a_j^I \\ &= \frac{1}{2\pi} K_{IJ} b^I, && (4.24) \end{aligned}$$

or

$$b^I = 2\pi (K^{-1})^{JI} \rho_J, \tag{4.25}$$

which implies that the physical meaning of coupling the gauge field with a matter current is to tie different magnetic fluxes b^I to different charges ρ_J through the K-matrix. Therefore, as in the single-component case, each quasiparticle in the multi-component system is a flux-charge composite.

4.4 Wilson Lines and Statistics

The gauge-invariant non-local objects of this theory are the Wilson lines. These operators describe the world line of the quasiparticles. The Wilson line associated with the world line of a quasiparticle l_I along a closed curve (or infinitely extended) in the x^i-direction is defined as

$$W_l[C] = \exp\left(i\int_C l_I a_i^I dx^i\right). \tag{4.26}$$

From the Wilson lines and the commutation relations between the gauge fields, it is possible to find the statistics of the quasiparticles in the system. To do that, first calculate the conjugate momentum Π_k^β as

$$\Pi_K^\beta = \frac{\partial L}{\partial(\partial_0 a_\beta^K)}. \tag{4.27}$$

From the Lagrangian in equation 4.13, the conjugate momentum in terms of the K-matrix becomes

$$\Pi_J^j = -\frac{1}{4\pi}K_{IJ}\varepsilon^{ij}a_i^I, \tag{4.28}$$

where A_μ was treated as a background field. We can use the commutation relation

$$[a_I^i(\vec{x}), \Pi_J^j(\vec{x}')] = i\delta_{IJ}\delta^{ij}\delta^2(\vec{x}-\vec{x}'), \tag{4.29}$$

where $i, j = 1, 2$ are spatial indices and $I, J = 1, \ldots, N$ is related to the dimension of the K-matrix, to present the commutator between the gauge fields as

$$[a_I^i(\vec{x}), a_J^j(\vec{x}')] = 2\pi i(K^{-1})_{IJ}\varepsilon^{ij}\delta^2(\vec{x}-\vec{x}'), \tag{4.30}$$

showing a non-trivial commutation relation between the gauge fields implying that they are a pair of conjugate fields. Notice the similarity to the commutating relation for the single-component CS theory in equation 3.3.

To proceed to find the statistics of the quasiparticles, let us define two Wilson lines as

$$W_l[C] = \exp\left(i\int_C l_I a_i^I dx^i\right) \tag{4.31}$$

$$W_m[C'] = \exp\left(i\int_{C'} m_J a_j^J dx^j\right), \tag{4.32}$$

where C and C' are curves extended in directions x^i and x^j, respectively. We want to find the phase that arises when we try to commute both operators, that is,

$$W_l[C]W_m[C'] = e^{i\theta}W_m[C']W_l[C]. \tag{4.33}$$

This can be solved by using the BCH theorem and calculating

$$\left[i\int_C l_I a_i^I dx^i, i\int_{C'} m_J a_j^J dx^j\right] = -2\pi i\int_C\int_{C'} l_I(K^{-1})^{IJ}m_J\varepsilon_{ij}\delta(x^i - x^j)dx^i dx^j. \tag{4.34}$$

Considering the indices $i = 1$ and $j = 2$, with $\varepsilon_{12} = 1$, one gets

$$\left[i \int_C l_I a_i^I dx^i, i \int_{C'} m_J a_j^J dx^j\right] = -2\pi i l_I (K^{-1})^{IJ} m_J \text{Link}(C, C'), \tag{4.35}$$

and substituting this result into the BCH theorem, we have

$$W_l[C] W_m[C'] = \exp\left[2\pi i\, l_I (K^{-1})^{IJ} m_J \,\text{Link}(C, C')\right] W_m[C'] W_l[C], \tag{4.36}$$

where the phase acquired when trying to exchange the Wilson operators is

$$e^{2\pi i\, l_I (K^{-1})^{IJ} m_J \,\text{Link}(C,C')}.$$

Assuming $\text{Link}(C, C') = 1$, the braiding angle on taking the quasiparticle $\mathbf{l}$ around the quasiparticle $\mathbf{m}$ once is given by

$$\theta_{lm} = 2\pi l_I (K^{-1})^{IJ} m_J, \tag{4.37}$$

while the exchange phase (mutual statistics) of a quasiparticle $\mathbf{l}$ is half of the braiding angle, that is,

$$\delta_l = \pi l_I (K^{-1})^{IJ} m_J. \tag{4.38}$$

4.5 Local Particles

We can search for local particles in the systems. These are the particles exhibiting trivial braiding to every other quasiparticle in the system, meaning that no quantum phase is generated when one particle encircles another. As in the single-component case, where the Wilson lines with $n = \kappa$ were transparent (trivial braiding to the other quasiparticles in the system), a clever guess for the multi-component theory would be that a quasiparticle excitation $\mathbf{m}$ with gauge charge

$$m_J = (K)_{IJ} \tilde{m}^I$$

is a local particle in the theory, where $\tilde{m}$ is a quasiparticle in the bulk.

We can see if this statement is true for this particular particle by using equation 4.37 for the braiding angle as

$$\theta_{lm} = 2\pi l_I \tilde{m}^I, \tag{4.39}$$

with $l, \tilde{m} \in \mathbb{Z}^N$. Since $l_I \tilde{m}^I \in \mathbb{Z}$, the braiding angle becomes

$$\theta_{lm} = 2\pi \mathbb{Z},$$

and therefore the statistics between $\mathbf{m}$ and any other quasiparticle $\mathbf{l}$ is an integer multiple of 2π. In other words, the quasiparticles $m_J = K_{IJ} \tilde{m}^J$ are indeed local to any other particle l^i, because the statistics is trivial. This means that these quasiparticles are "gauge-invariant" degrees of freedom, that is, they do not depend on the gauge choice. Thus, they are physical observables, unlike the emergent quasiparticles with non-trivial statistics, which are only treated effectively. The electrons in a fractional quantum Hall state, for example, are physical local "gauge-invariant" degrees of freedom [49].

Local particles have integer multiples of the electron charge e. We can see this explicitly using the equation 4.19 for the charge m and substituting the local form $m_j = (K)_{kj}\tilde{m}^k$ as

$$\begin{aligned} q_m &= t_I(K^{-1})^{IJ}m_J \\ &= t_I(K^{-1})^{IJ}K_{KJ}\tilde{m}^K \\ &= t_I\tilde{m}^I \in \mathbb{Z}, \end{aligned} \tag{4.40}$$

with $(K^{-1})^{IJ}K_{KJ} = \delta^I_K$. The charge in this equation is defined in units of e and, therefore, the charge associated with the particle $\mathbf{m}$ is indeed an integer multiple of e. We can say that $\mathbf{m}$ is a local composite of electrons [12].

4.6 Equivalences

With the definition of local particles in the system, it is possible to introduce the concept of equivalence between quasiparticles in our theory. Consider $\mathbf{l}'$ and $\mathbf{l}''$ quasiparticles in the bulk. They are equivalent if they differ by an addition of local quasiparticles, that is, if

$$\mathbf{l}' - \mathbf{l}'' = K\mathbf{l} \qquad ; \qquad \mathbf{l},\ \mathbf{l}',\ \mathbf{l}'' \in \mathbb{Z}^N,$$

in the sense that they share the same topological properties of braiding and exchange statistics. They might not share the same charge, since this is not a topological characteristic [30].

As already seen, a generic quasiparticle $\mathbf{l}$ belongs to the lattice $\Gamma = \mathbb{Z}^N$. As for the local particles, they belong to the dual lattice $\Gamma^* = K\mathbb{Z}^N$. Therefore, distinct quasiparticles are distinguished through equivalence classes of $\mathbf{l}$ modulo the local vectors $K\mathbb{Z}^N$, that is, through the quotient group

$$\mathscr{A} = \frac{\Gamma}{\Gamma^*} = \frac{\mathbb{Z}^N}{K\mathbb{Z}^N}.$$

Let us analyze how the elements of the K-matrix relate to the statistics. For that, consider $l = K\tilde{l}$ a local vector. Then, the expression for the statistics is

$$\begin{aligned} \delta_l &= \pi l^T K^{-1} l \\ &= \pi \tilde{l}^T K \tilde{l}, \end{aligned} \tag{4.41}$$

because $KK^{-1} = \mathbb{I}$. We can divide this expression using the index notation as

$$\begin{aligned} \delta_l &= \pi K_{II}(\tilde{l}^I)^2 + \sum_{I<J} 2\pi \tilde{l}_I K_{IJ} \tilde{l}_J \\ &= \pi K_{II}(\tilde{l}^I)^2 + 2\pi\mathbb{Z}. \end{aligned} \tag{4.42}$$

Notice that if all the diagonal elements of the K-matrix are even integers ($K_{II} = 2\mathbb{Z}, \forall I$, then $\delta_l \in 2\pi\mathbb{Z}$ and therefore the physical degrees of freedom have bosonic statistics

$$\delta_l = 0 \bmod 2\pi,$$

and the Lagrangian associated with this K-matrix describes a bosonic system with $|\text{Det}K|$ independent Wilson lines [91]. On the other hand, if at least one diagonal element of the K-matrix is an odd integers ($K_{II} = 2\mathbb{Z}+1$ for some I), then $\delta_l \in \pi\mathbb{Z}$ and therefore there are one or more (depending on the number of odd diagonal elements) fermionic physical degrees of freedom with fermionic statistic

$$\delta_l = \pi \bmod 2\pi,$$

and, because of that, the system has local fermionic operators and, thus, it requires a spin structure. In this case, there are $2|\text{Det}K|$ independent Wilson lines [91].

Since K is a symmetric matrix, it can be diagonalized via

$$K = UDU^T \qquad \text{or} \qquad K_{IJ} = U_{IM}D^{MN}U_{NJ}, \tag{4.43}$$

where U is an orthogonal matrix, i.e., $U \in O(N)$ and $|\text{Det}(U)| = 1$. The gauge fields a^i_μ also transforms under U as

$$a^j_\mu = U^{ij}(\tilde{a}_\mu)_i. \tag{4.44}$$

By acting this transformation under the CS term in the Lagrangian, one gets

$$\begin{aligned} L_{CS} &= \frac{1}{4\pi}\varepsilon^{\mu\nu\rho}a^I_\mu K_{IJ}\partial_\nu a^J_\rho \\ &= \frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\tilde{a}^J_\mu D_{JI}\partial_\nu \tilde{a}^I_\rho, \end{aligned} \tag{4.45}$$

where D_{JI} is the diagonal matrix of eigenvalues of K, that is,

$$D = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & k_N \end{bmatrix},$$

and using this procedure, it is possible to decouple the theory into N independent CS terms.

Each term in the diagonal matrix D has ground-state degeneracy (GSD) $|k_I|$ when the theory is placed on a torus. Thus, the whole theory has a ground-state degeneracy given by

$$\begin{aligned} GSD &= |k_1||k_2|\ldots|k_n| \\ &= |\text{Det}(D)|. \end{aligned}$$

Since

$$\begin{aligned}|\text{Det}(K)| &= |\text{Det}(UDU^T)| \\ &= |\text{Det}(U)||\text{Det}(D)||\text{Det}(U^T)| \\ &= |\text{Det}(D)|,\end{aligned}$$

the ground-state degeneracy of the CS $U(1)^N$ theory on the torus is equal to the absolute value of the K-matrix determinant $|\text{Det}(K)|$. Since the definition of two quasiparticles l and l' is the same if they accumulate the same braiding phase when moved adiabatically around any other quasiparticle l'', the system has $|\text{Det}(K)|$ different quasiparticles, which is the same number as the ground-state degeneracy on the torus. Therefore, there are $|\text{Det}(K)|$ elements in the set $\mathcal{A}$.

Now, suppose a change of basis in the space of the gauge field by redefining

$$a \to G^T a, \tag{4.46}$$

where $G \in GL(N, \mathbb{Z})$ is a $N \times N$ matrix with integer entries and unimodular ($|\text{Det}(G)| = 1$). This matrix is required to be unimodular because the transformation on the gauge field a has to be invertible and preserve the volume of the lattice. If we had $|\text{Det}(G)| \neq 1$, or fractional entries, the quantization conditions of charges would no longer be valid, and the fields a would no longer be compact. Under the transformation in equation 4.46, the Lagrangian in equation 4.2 becomes

$$\begin{aligned}L' &= \frac{1}{4\pi}\varepsilon^{\mu\nu\rho} a_\mu^T G K G^T \partial_\nu a_\rho - \frac{1}{2\pi}\varepsilon^{\mu\nu\rho} t^T A_\mu \partial_\nu (G^T a_\rho) \\ &= \frac{1}{4\pi}\varepsilon^{\mu\nu\rho} a_\mu^T G K G^T \partial_\nu a_\rho - \frac{1}{2\pi}\varepsilon^{\mu\nu\rho} (Gt)^T A_\mu \partial_\nu a_\rho.\end{aligned} \tag{4.47}$$

Let us summarize some previous results as

$$\nu = t^T K^{-1} t \tag{4.48}$$

$$q_l = t^T K^{-1} l \tag{4.49}$$

$$\theta_{lm} = 2\pi l^T K^{-1} m \tag{4.50}$$

$$\delta_l = \pi l^T K^{-1} l. \tag{4.51}$$

Notice that these quantities and the GSD are invariant under the following transformations

$$K \to \tilde{K} = GKG^T \quad ; \quad t \to \tilde{t} = Gt \quad ; \quad l \to \tilde{l} = Gl, \tag{4.52}$$

with $G \in GL(N, \mathbb{Z})$. These are the exact quantities in the transformed Lagrangian under the basis change on the gauge fields in equation 4.47. That is, using the transformations in equation 4.52, one gets

$$L[a, A] = \frac{1}{4\pi}\varepsilon^{\mu\nu\rho} a_\mu^T \tilde{K} \partial_\nu a_\rho - \frac{1}{2\pi}\varepsilon^{\mu\nu\rho} \tilde{t}^T A_\mu \partial_\nu a_\rho, \tag{4.53}$$

which is a Chern-Simons theory associated with the transformed $\tilde{K}$-matrix.

That means that if two different matrices K and $\tilde{K}$ are related to each other by the transformations in equation 4.52, they describe the same topological phase with the same set of quasiparticle excitations (superselection sectors), despite being labeled in different ways. Therefore, these matrices are equivalent. In other words, the correspondence between K-matrices and topological phases is not a bijection (one-to-one), and both Lagrangians L and L' are equivalent, as they have the same topological characteristics [46, 49]. These equivalences of theories are known as Lagrangian or classical symmetries [91].

The equivalence between the phases described by K and $\tilde{K}$ can be interpreted as a simple basis change of the gauge fields, where a_μ and $G^T a_\mu$ describe the same degrees of freedom. Another interpretation is that $G^T a_\mu$ is constructed from a_μ as its dual gauge field that characterizes a different degree of freedom, but describes the same physical system [64].

4.6.1 Example

Consider the Lagrangian for a Laughlin state at filling $\nu = 1/(2n+1)$ $(n = 0, 1, \dots)$ in contact with a topological s-wave superconductor as

$$L = \frac{2n+1}{4\pi}\alpha \wedge d\alpha - \frac{1}{2\pi}a \wedge d\alpha + \frac{1}{\pi}a \wedge db, \tag{4.54}$$

which can be written in the K-matrix formulation as

$$L = \frac{1}{4\pi}K_{IJ}\beta^I \wedge d\beta^J, \tag{4.55}$$

with $\beta = (\alpha, a, b)^T$ and

$$K = \begin{bmatrix} 2n+1 & -1 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}. \tag{4.56}$$

This K- matrix allows us to return to the expanded formulation of the theory in Equation 4.54.

We want to verify if there is another theory that is equivalent to this one. To do that, we need to find a G-matrix with $\text{Det}(G) = 1$, which will allow us to find the associated equivalent theory. Consider the following G-matrix

$$G = \begin{bmatrix} 2 & 2(2n+1) & 1 \\ 1 & n & 0 \\ -1 & -(n+1) & 0 \end{bmatrix}, \tag{4.57}$$

which has $|\text{Det}(G)| = 1$ and therefore $G \in GL(3, \mathbb{Z})$. This is a G-matrix that can transform the fields to obtain a new, but equivalent, Chern-Simons theory.

Using the transformation matrix in equation 4.57, one can find the corresponding equivalent matrix $\tilde{K}$ as

$$K \to K' = GKG^T \tag{4.58}$$

and calculating this product of matrices, one gets

$$K' = \begin{bmatrix} 8n+4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1. \end{bmatrix} \tag{4.59}$$

Since K and K' are equivalent, the topological features of both theories specified by K and K' are the same. Notice that K' describes bosonic "Laughlin" states at filling $\nu = 1/(8n+4)$ and a fermionic sector with two fermions. Notice that the fermionic sector given by σ_z is trivial from the topological view since it does not affect the topology of the system. Therefore, this sector can be ignored in the IR (low-energy) effective theory, since these modes can be gapped or lifted to higher energies. Thus, we can say that K' describes "Laughlin" states at $\nu = 1/(8n+4)$ modulo two fermion modes [30].

There is another way of finding this equivalence between the theories of Laughlin state at $\nu = 1/(2n+1)$ in contact with an s-wave superconductor and a "Laughlin" state at $\nu = 1/(8n+4)$ by integrating the field a_μ out in equation 4.54

$$L = \frac{2n+1}{4\pi}\alpha \wedge d\alpha - \frac{1}{2\pi}a \wedge d\alpha + \frac{1}{\pi}a \wedge db. \tag{4.60}$$

Finding the equation of motion for a_μ,

$$\frac{\partial L}{\partial a} = 0 \iff d\alpha = 2db \implies \alpha = 2b, \tag{4.61}$$

which is a constraint for the fields α and b. Substituting the constraint in equation 4.61 into the Lagrangian, one gets

$$L = \frac{8n+4}{4\pi}b \wedge db, \tag{4.62}$$

which is the "Laughlin" state at filling $\nu = 1/(8n+4)$.

As a result of the constraint in equation 4.61, the Wilson lines also have a constraint associated, which is

$$e^{i\int a} = e^{i\int 2b}. \tag{4.63}$$

Recall that a fermionic theory has $2k$ independent lines. Therefore, the theory described by

$$K = \begin{bmatrix} 2n+1 & -1 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \tag{4.64}$$

has $4n+2$ independent lines while the theory with $\nu = 8n+4$ has $8n+4$ independent lines, as represented in Figure 11.

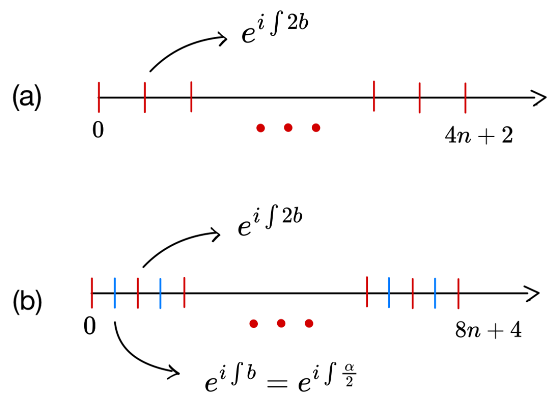


Figura 11 – Illustration of the number of independent lines in (a). the original theory with $\nu = 1/2n+1$ and (b). the system in contact with an s-wave superconductor with $\nu = 1/8n+4$.

The set of distinct quasiparticles for the "Laughlin" state at $\nu = 1/(8n+4)$ is given by

$$\mathscr{A} = \{m^{8n+4} \equiv 1, m, m^2, m^3, \dots, m^{8n+3}\},$$

where was defined $1 \equiv m^{8n+4}$, $\varepsilon \equiv m^2$ (Laughlin quasiparticle) and $\Psi = m^{4n+2}$ (electron). Defining the statistics of the m^k quasiparticle as

$$\delta_{m^k} = \frac{\pi k^2}{8n+4} \mod 2\pi, \tag{4.65}$$

the statistics for

$$\delta_{m^{8n+4}} = \delta_1 = \pi \mod 2\pi \tag{4.66}$$

$$\delta_{m^2} = \delta_\epsilon = \frac{\pi}{2n+1} \mod 2\pi \tag{4.67}$$

$$\delta_{m^{4n+2}} = \delta_\Psi = \pi(2n+1) \mod 2\pi = \pi \mod 2\pi \tag{4.68}$$

we see that $m^2 = \epsilon$ corresponds to a quasiparticle in a Laughlin state with $\nu = 2n+1$ and $m^{4n+2} = \Psi$ has fermionic statistics. In addition, the fusion rule is defined as

$$m^k \times m^l = m^{k+l \mod 8n+4} \tag{4.69}$$

In the particular case with $n = 1$ in the example above, the filling factor is $\nu = 1/12$, and the system has 12 different quasiparticles, represented as elements in the set

$$\mathscr{A} = \{m^{12} \equiv 1, m, m^2 \equiv \varepsilon, \dots m^6 \equiv \Psi, \dots, m^{11}\}, \tag{4.70}$$

with the fusion group

$$m^k \times m^l = m^{k+l \mod 12}, \tag{4.71}$$

and braiding statistics

$$\delta_{m^k} = \frac{\pi k^2}{12} \mod 2\pi \tag{4.72}$$

This system is a Laughlin state at filling $\nu = 1/3$ in contact with an s-wave superconductor, being equivalent to a state with filling $\nu = 1/12$.

4.7 Anyonic Symmetries

An anyonic symmetry is a permutation of anyons that preserves the braiding exchange and the fusion rules. These symmetries are related to G matrices that leave the Lagrangian invariant under such transformations.

To study anyonic symmetries in our system, let us analyze the transformations that leave the Lagrangian invariant. To do that, consider only the CS term (i.e., turning off the background field A_μ) as in equation 4.20.

Define the subset of all G-matrices that act on K by leaving this matrix identically unchanged as the group of automorphisms of K

$$\text{Aut}(K) = \{G \in GL(N, \mathbb{Z}); GKG^T = K\}, \tag{4.73}$$

which means that the transformations $G \in \text{Aut}(K)$ are symmetries of the Lagrangian, since applying $K \to GKG^T$ and making a change of basis $a \to G^T a$; $l \to Gl$ maintain the Lagrangian invariant. However, notice that some G-matrices act trivially on the quasiparticle labels $\mathbf{l}$ in $\mathscr{A}$, which means that they preserve this vector up to the addition of local particles of the form $K\mathbb{Z}^N$.

The set of these special matrices G_0 that act trivially on the quasiparticle's labels is a normal subgroup of $\text{Aut}(K)$ of inner isomorphisms, that is,

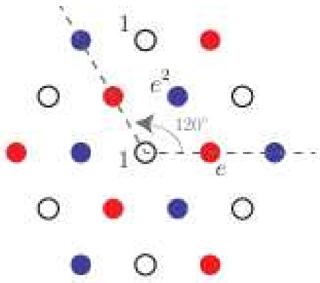
$$\text{Inner}(K) = \{G_0 \in \text{Aut}(K); [G_0 l] = [l] = l + K\mathbb{Z}^N\}, \tag{4.74}$$

where $[\,.\,]$ denotes the equivalence class of the argument, meaning that G_0 preserves the anyons' equivalent classes.

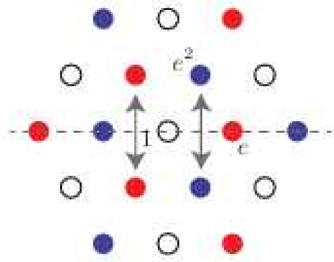
Since the interest lies in the non-trivial anyons' relabelings, we take off the trivial transformations through the construction of the quotient group

$$\text{Outer}(K) = \frac{\text{Aut}(K)}{\text{Inner}(K)}, \tag{4.75}$$

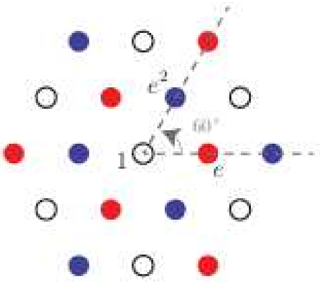
which is the group of anyonic symmetries of the topological phase characterized by the K-matrix. Therefore, the transformation $G \in \text{Outer}(K)$ is an anyonic symmetry, which respects the invariance of braiding exchange and fusion rules, as we will see in the example below.



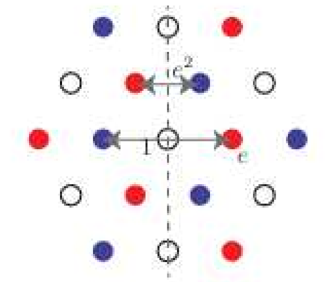
(a) Inner Automorphism where the lattice is rotated through 120°



(b) Inner automorphism corresponding to reflection about the horizontal dotted grey line



(c) Outer Automorphism where the lattice is rotated through 60°



(d) Outer automorphism corresponding to reflection about the vertical dotted grey line

Figura 12 – Examples of automorphisms in a lattice composed of three types of particles $(1, e, e^2)$, where the figures show examples of (a). inner automorphisms by rotation, (b). inner automorphisms by rotation, (c). outer automorphisms by rotation and (d). outer automorphisms by reflection. From Ref. [5].

A simple and illustrative example of inner and outer automorphisms in a lattice composed of three different particles, $1, e, e^2 \in \mathbb{Z}/K\mathbb{Z}^2$, is shown in Figure 12. The inner automorphisms in (a) and (b) are related to rotations by 120^o from the horizontal axis and reflection over the horizontal axis, where the lattice remains the same under these transformations, that is, rotation by this angle and reflection over the x-axis act trivially in the lattice. On the other hand, the outer automorphisms in (c) and (d) are related to rotations by 60^o from the horizontal axis and reflection over the vertical axis, where these transformations exchange/permute e and e^2, implying an anyonic symmetry.

4.7.1 Example

Let us analyze the properties of anyonic symmetries with an example. Remember the case of an FQH system at filling $\nu = 1/8n{+}4$, where the set of all distinct quasiparticles is

$$\mathscr{A} = \{1 = m^{8n+4}, m, \ldots, m^{8n+3}\}. \tag{4.76}$$

We want to find all the permutation operations $\mathcal{P}$ such that the map $\mathscr{A} \to \mathcal{P}\mathscr{A}$ preserves the fusion algebra and the braiding statistics. For that, we need to define the action of the permutation operator $\mathcal{P}$ on an element $m^k \in \mathscr{A}; k = 1, 2, \ldots, 8n+4$ as

$$\mathcal{P}m^k = m^{pk}; \qquad p \in \mathbb{Z}^+. \tag{4.77}$$

The assumption of invariance of the braiding statistics leads to a constraint on the values of p. We can see this explicitly by considering the invariance of the braiding under the action of the permutation operation $\mathcal{P}$ as

$$\delta_{m^k} = \delta_{\mathcal{P}m^k} \bmod 2\pi. \tag{4.78}$$

From the expression of the braiding statistics of a quasiparticle m^k in equation 4.65 with $K^{-1} = 1/8n+4$ and by renaming $2n+1 \to n$, the statistics become

$$\delta_{m^k} = \pi\frac{k^2}{4n} \to \delta_{\mathcal{P}m^k} = \pi\frac{p^2k^2}{4n} \bmod 2\pi = \pi\frac{k^2}{4n} + \pi\frac{(p^2-1)k^2}{4n} \bmod 2\pi. \tag{4.79}$$

The braiding statistics of m^k and $\mathcal{P}m^k$ are the same if

$$\begin{aligned} p^2 - 1 = &\bmod 8n & (4.80)\\ \implies p^2 &= 1 \bmod 8n \\ \implies p^2 &= 1 \bmod 16n+8, & (4.81) \end{aligned}$$

where in the last line we renamed back $n \to 2n+1$. This is the resulting constraint in the values of p so that the operation $\mathcal{P}$ is an anyonic symmetry. In other words, a transformation $m^k \to m^{pk}$ performed by a permutation operator $\mathcal{P}$ is an anyonic symmetry if p obeys the constraint in equation 4.81.

As all permutations are bijection functions $\mathcal{P} : \mathscr{A} \to \mathscr{A}$, the integers p satisfying the constraint in equation 4.81 are precisely those that are coprime to $8n+4$, that is,

$$\gcd(p, 8n+4) = 1, \tag{4.82}$$

where gcd denotes "greatest common divisor". Therefore, only the odds p's that are mutually prime with $8n+4$ are candidates for anyonic symmetries.

4.7.2 Case $n = 1$

Let us further analyze the state with $\nu = 1/8n+4$ with $n = 1$, that is, the state at filling fraction $\nu = 1/12$ described by a single level $K = 12$. Recall that the set of distinct quasiparticles is

$$\mathcal{A} = \{m^{12} = 1, m, m^2, \ldots, m^{11}\} \tag{4.83}$$

For this case, the realizable p that are candidates to be anyonic symmetries are $p = 5, 7,$ and 11 since

$$\gcd(5, 12) = \gcd(7, 12) = \gcd(11, 12) = 1.$$

The action of these anyonic symmetries is

$$m \to m^5 \tag{4.84}$$

$$m \to m^7 \tag{4.85}$$

$$m \to m^{11}. \tag{4.86}$$

In addition, recall that, for this topological state, the statistics is given by

$$\delta_{m^k} = \frac{\pi k^2}{12} \mod 2\pi \tag{4.87}$$

For example, the quasiparticle $m^{12} = 1$ has statistics

$$\delta_{m^{12}} = \frac{\pi}{12} \mod 2\pi$$

and the quasiparticle m^2 has statistics

$$\delta_{m^2} = \frac{4\pi}{12} \mod 2\pi = \frac{\pi}{3} \mod 2\pi,$$

which are fractional braiding statistics, that is, both these particles are anyons in the system.

We want to find the G-matrices that lead to the candidates for anyonic symmetries in our system related to $p = 5, 7,$ and 11. For that, we want to find the G-matrices that satisfy the automorphism

$$G \in GL(1, \mathbb{Z}); \quad GKG^T = K; \quad |\text{Det}K| = 1, \tag{4.88}$$

with $K = 12$. Notice that $G = \pm 1$ satisfies this condition. However, $G = +1$ acts trivially on the anyons of our system, so we can exclude it from the set of realizable matrices G. Thus, only $G_{11} = -1$ is in the set Outer(K), which is the anyonic symmetries group.

The action of $G_{11} = -1$ on m^k is defined as

$$Gm^k = m^{-k \mod 12} = m^{12-a}, \tag{4.89}$$

with self-statistics

$$\begin{aligned} \delta_{Gm^k} = \delta_{m^{-k}} = \frac{\pi(-k)^2}{12} \quad \mod 2\pi = \frac{\pi k^2}{12} \quad \mod 2\pi \\ = \delta_{m^k}, \end{aligned} \tag{4.90}$$

and therefore the G-transformation does not change the exchange phase of m^k.

Recall that an anyonic symmetry is a permutation operator that preserves braiding statistics and fusion rules of the quasiparticles in the system. We have already seen that the permutation operation related to G_{11} preserves the braiding statistics. Then, we have to prove that this transformation also preserves the quasiparticle fusion rule. Let us analyze the fusion rule under the application of the transformation $G_{11} = -1$ as

$$\begin{aligned} G(m^k \times m^l) &= Gm^{k+l \mod 12} = m^{-(k+l) \mod 12} \\ &= m^{-k} \times m^{-l} \\ &= Gm^k \times Gm^l, \end{aligned} \tag{4.91}$$

which shows that the fusion rule is preserved.

Since the braiding exchange and the fusion rule are preserved under the application of G, then $G_{11} = -1$ is a realizable anyonic symmetry. This G-transformation corresponds to $p = 11$, since $m^{-1 \mod 12} = m^{11}$. However, the G-transformations related to $p = 5$ and $p = 7$ are valid candidates to be anyonic symmetries, but $G_{11} = -1$ does not represent these additional symmetries.

4.8 Stable equivalence

However, the group of outer isomorphisms related to anyonic symmetries presented in the last section may not account for all possible anyonic symmetries in the system. In this case, it is necessary to consider another type of equivalence, known as stable equivalence, which will be explored further in this chapter.

To understand the concept of stable equivalence, we must first understand certain types of matrices that represent a topologically trivial sector. These matrices are called "trivial" in the sense that they have a set of degrees of freedom that does not affect statistics and GSD of the system, and also they do not carry charge. For example,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.92}$$

is a bosonic (even diagonal entries) trivial sector with $|\mathrm{Det}\, \sigma_x| = 1$, which describes trivial bosonic local modes. On the other hand,

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.93}$$

is a fermionic (odd diagonal entries) trivial sector with $|\mathrm{Det}\ \sigma_z| = 1$, which describes trivial fermionic local modes.

Given a K-matrix, it is possible to extend it by adding trivial decoupled sectors to it. In this case, the enlarged K'-matrix is stably equivalent to the original K-matrix, that is,

$$K \sim K' = K \oplus \sigma_x \tag{4.94}$$

$$K \sim K' = K \oplus \sigma_x, \tag{4.95}$$

where the notation $\sim$ denotes that the matrices are stably equivalent to each other.

In the context of anyonic symmetries, in most cases, it is possible to find all representative elements from K corresponding to each realizable anyonic symmetry. However, there are cases when this is not possible. Then, it becomes necessary to consider an enlarged K'-matrix stably equivalent to the original one to find all possible anyonic symmetries related to each G-matrix transformation.

4.8.1 Example

As already seen, the system at filling $\nu = 1/12$ has an anyonic symmetry $m \to m^{-1} = m^{11}$ related to $p = 11$ with $G_{11} = -1$. However, the system signals two other anyonic symmetries $m \to m^5$ and $m \to m^7$ that cannot be performed by G_{11}. Therefore, it is necessary to use the concept of stable equivalence to enlarge the K-matrix and search for the other two additional anyonic symmetries.

Firstly, let us verify that the permutations associated with $p = 5$ and $p = 7$ are also anyonic symmetries. To do that, let us use the braiding statistics in equation 4.65 to calculate

$$\begin{aligned} \delta_{m^5} &= \frac{\pi(5)^2}{12} = \frac{25\pi}{12} = \frac{24\pi}{12} + \frac{\pi}{12} \mod 2\pi \\ &= \frac{\pi}{12} \mod 2\pi, \end{aligned} \tag{4.96}$$

and

$$\begin{aligned} \delta_{m^7} &= \frac{\pi(7)^2}{12} = \frac{49\pi}{12} = \frac{48\pi}{12} + \frac{\pi}{12} \mod 2\pi \\ &= \frac{\pi}{12} \mod 2\pi \end{aligned} \tag{4.97}$$

to notice that the braiding statistics are the same as δ_m. This signals that the system may have two more anyonic symmetries $m \to m^5$ and $m \to m^7$ not realizable from $G_{11} = -1$.

Let us use the concept of stable equivalence to construct an enlarged K'-matrix from $K = 12$, adding a bosonic trivial sector to it, that is,

$$K' = K \oplus \sigma_x \tag{4.98}$$

$$= \begin{bmatrix} 12 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \qquad (K')^{-1} = \begin{bmatrix} 1/12 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{4.99}$$

where σ_x is the x-Pauli matrix and $(K')^{-1}$ is the inverse of K'. As a result of the stable equivalence between both matrices, $K \sim K'$, they describe the same topological state.

Consider the following G-matrix

$$G_5 = \begin{bmatrix} 5 & -12 & 12 \\ 1 & -2 & 3 \\ -1 & 3 & -2 \end{bmatrix} \tag{4.100}$$

and $m^T = (1, 0, 0)$. Notice that the first line is composed of $p = 5$ and integers $\mod 2\pi$. The action of G_5 over $(m^k)^T = (k, 0, 0)$ is given by

$$\begin{aligned} G_5 m^k &= \begin{bmatrix} 5 & -12 & 12 \\ 1 & -2 & 3 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5k \\ k \\ -k \end{bmatrix} = m', \end{aligned} \tag{4.101}$$

and the braiding statistics associated with m^k is, in generalized form,

$$\delta_{m^k} = \pi (m^k)^T K^{-1} m^k, \tag{4.102}$$

with $(m^k)^T = (k, 0, 0)$.

Firstly, let us verify if the transformation matrix G preserves the statistics of a quasiparicle m. The braiding statistics of the resulting quasiparticle associated with the action of G_5 over m is given by

$$\begin{aligned} \delta_{G_5 m} &= \pi (G_5 m)^T K^{-1} (G_5 m) \\ &= \pi \begin{bmatrix} 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/12 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} \\ &= \frac{25\pi}{12} \mod 2\pi = \delta_{m^5} \\ &= \frac{\pi}{12} \mod 2\pi, \end{aligned} \tag{4.103}$$

which is the same braiding statistics of $m^T = (1, 0, 0)$ and $(m^5)^T = (5, 0, 0)$. Therefore, the braiding statistics is preserved under the action of G_5 since $\delta_{G_5 m} = \delta_m$. Furthermore, it is possible to infer that the matrix G_5 implements the transformation $m \to m^5$.

To prove that G_5 is an anyonic symmetry, it remains to prove that this transformation also preserves the quasiparticle's fusion rule. The action of G_5 under the fusion rule is given by

$$
\begin{aligned}
G_5[m^k \times m^l] &= G_5 m^{k+l} \\
&= \begin{bmatrix} 5 & -12 & 12 \\ 1 & -2 & 3 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} k+l \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 5(k+l) \\ k+l \\ -(k+l) \end{bmatrix} \\
&= \begin{bmatrix} 5(k \\ k \\ -k) \end{bmatrix} + \begin{bmatrix} 5l \\ l \\ -l \end{bmatrix} \\
&= G_5 m^k \times G_5 m^l,
\end{aligned} \tag{4.104}
$$

which shows that the fusion rule is preserved and the transformation $m \to m^5$ via G_5 is an anyonic symmetry.

We can use the same arguments to show that the following G-matrix transformation

$$
G_7 = \begin{bmatrix} 7 & 12 & -24 \\ 1 & 2 & -3 \\ -2 & -3 & 8 \end{bmatrix} \tag{4.105}
$$

realize the anyonic symmetry $m \to m^7$. This transformation acts on $(m^k)^T = (k, 0, 0)$ as

$$
\begin{aligned}
G_7 m^k &= \begin{bmatrix} 7 & 12 & -24 \\ 1 & 2 & -3 \\ -2 & -3 & 8 \end{bmatrix} \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 7k \\ k \\ -2k \end{bmatrix} = m'.
\end{aligned} \tag{4.106}
$$

It is enough to show that

$$
\delta_{G_7 m^k} = \delta_{m^k} \tag{4.107}
$$

$$
G_7(m^k \times m^l) = Gm^k \times Gm^l \tag{4.108}
$$

to prove that the G_7-transformation is an anyonic symmetry.

Therefore, the enlarged K'-matrix obtained by stable equivalence enabled us to find the two additional anyonic symmetries in our system, related to $p = 5$ and $p = 7$.

5 $U(1)^N$ CHERN-SIMONS EDGE THEORY

In this chapter, we develop the edge theory of the $U(1)^N$ Chern-Simons theory by assuming that the bulk $\mathcal{M}$ has a boundary $\partial_{\mathcal{M}}$. Further, we will analyze perturbations on this edge and the conditions to gap the edge mode excitations.

5.1 Edge Lagrangian

The bulk CS action in 2+1 dimensions defined in a manifold $\mathcal{M}$ is given by

$$S_{\text{bulk}} = \int_{\mathcal{M}} d^3x \frac{1}{4\pi} K_{IJ} \varepsilon^{\mu\nu\rho} a^I_\mu \partial_\nu a^J_\rho. \tag{5.1}$$

This action is gauge-invariant up to boundary terms. Therefore, S_{bulk} is strictly invariant only if the manifold $\mathcal{M}$ has no boundary. In the case where $\mathcal{M}$ has a boundary $\partial\mathcal{M}$, boundary terms on the edge may arise from the action. These edge degrees of freedom cannot be dismissed as gauge redundancies by fixing the gauge. Thus, it is expected that there will be physical (dynamical) modes in the edge theory. It is possible to derive the CS edge theory in the K-matrix formalism using bosonization in 1+1 dimensions.

Consider the manifold $\mathcal{M}$ decomposed as

$$\mathcal{M} = \mathbb{R} \times \Sigma,$$

with

$$\Sigma = (-\infty, 0] \times \mathbb{R},$$

where $x^1 \in (-\infty, 0]$, so that there is a physical boundary at $x^1 = 0$, as shown in Figure 13, where we are considering the bulk in the bottom semi-plane. Also, assume the pure gauge configuration in the bulk as

$$a^I_\mu = \partial_\mu \phi^I, \tag{5.2}$$

with $\phi^i = \phi^i(x^1, x^2)$ and the scalar field being compact, that is, $\phi^i \sim \phi^i + 2\pi n^i$; $n^i \in \mathbb{Z}$.

To proceed with bosonization, we need to consider the commutation relation between the gauge field a^i_I as

$$[a^i_I(\vec{x}), a^j_J(\vec{x}')] = 2\pi i (K^{-1})_{IJ} \varepsilon^{ij} \delta^2(\vec{x} - \vec{x}'). \tag{5.3}$$

By substituting the pure gauge configuration into the commutation relation, one has

$$[\partial_i \phi^I(\vec{x}), \partial_j \phi^J(\vec{y})] = 2\pi i (K^{-1})^{IJ} \varepsilon_{ij} \delta(\vec{x^1} - \vec{y}^1) \delta(\vec{x^2} - \vec{y}^2). \tag{5.4}$$

Integrating this relation over x^1 in the interval $x^1 \in (-\infty, 0]$ and fixing $i = x^1 = 1$, one gets

$$[\phi^I(x^2), \partial_{y^2} \phi^J(y^2)] = 2\pi i (K^{-1})^{IJ} \delta(x^2 - y^2), \tag{5.5}$$

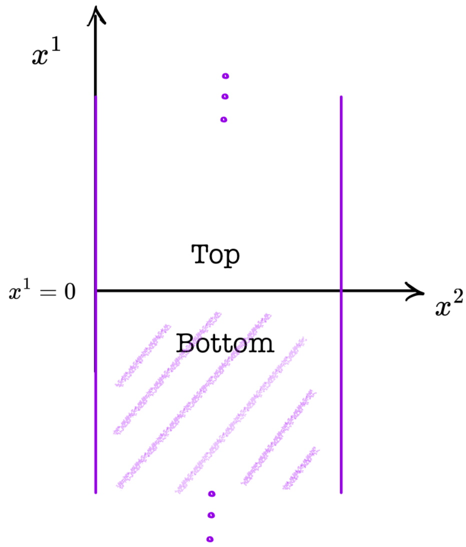


Figura 13 – Illustration of the bottom edge of a $U(1)^N$ Chern-Simons theory. In the figure, the bulk corresponds to the x_1-x_2 plane where $x_1 \in (-\infty, 0]$ (represented in the pink-shaded region) while the edge is located at $x_1 = 0$.

and assuming $x^2 = x$ and $y^2 = y$ to simplify, then

$$[\phi^I(x), \partial_y \phi^J(y)] = 2\pi i (K^{-1})^{IJ} \delta(x - y), \tag{5.6}$$

which is the Kac-Moody algebra for the edge fields ϕ^I. This expression resembles the commutation relation for a chiral boson field.

Sometimes, it is convenient to write the commutator in terms of the fields only. To do that, integrate this commutation relation over y to get

$$[\phi^I(x), \phi^J(y)] = \pi i (K^{-1})^{IJ} \text{sign}(x - y), \tag{5.7}$$

showing that the edge fields ϕ^I and ϕ^J form a pair of conjugate variables.

To find the Lagrangian of the edge theory, we first need to find the conjugate momentum. For that, suppose that $\partial_y \phi^J(y)$ is a function of $\Pi^J(y)$, i.e.,

$$\partial_y \phi^J(y) = A^{IJ} \Pi_I(y), \tag{5.8}$$

where A^{IJ} is a coefficient to be determined. Applying this relation to the commutator in equation 5.6, one has

$$A^{IJ} = 4\pi (K^{-1})^{IJ}, \tag{5.9}$$

and therefore the conjugate momentum is

$$\Pi_I(y) = \frac{1}{4\pi} K_{IJ} \partial_y \phi^J(y). \tag{5.10}$$

The conjugate momentum relates to the Lagrangian as

$$\Pi_I = \frac{\partial L}{\partial(\partial_t \phi^I)}. \tag{5.11}$$

Then, the Lagrangian term that gives rise to this momentum is given by

$$L^0_{\text{edge}} = \frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J. \tag{5.12}$$

However, notice that this Lagrangian has no propagating degrees of freedom, since it leads to a null Hamiltonian. Thus, as we are assuming that there are dynamical degrees of freedom in the edge, we must add a propagating term compatible with the symmetries of the system.

An adequate choice to add to the Lagrangian as a propagating term would be

$$\begin{aligned} L_{\text{edge}} &= L^0_{\text{edge}} + L^1_{\text{edge}} \\ &= \frac{1}{4\pi} K_{IJ} \partial_t \phi^I \partial_x \phi^J - \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J, \end{aligned} \tag{5.13}$$

where V_{IJ} is a positive-definite $N \times N$ symmetric matrix determined by the microscopic characteristics of the system, which is related to the absolute value of the velocities of the edge modes and interactions between the edge modes. This matrix is positive-definite because the corresponding Hamiltonian density

$$H = \int dx \frac{1}{4\pi} V_{IJ} \partial_x \phi^I \partial_x \phi^J \tag{5.14}$$

must be bounded from below, that is, it must also be positive-definite (the corresponding eigenvalues of V_{IJ} are all strictly positive).

The equation of motion associated with the edge theory can be obtained from the Euler-Lagrange equation for the field ϕ^K

$$\partial_\beta \frac{\partial \mathcal{L}_{CS}}{\partial(\partial_\beta\, \phi^K)} - \frac{\partial \mathcal{L}_{CS}}{\partial \phi^K} = 0, \tag{5.15}$$

as

$$(K_{IJ}\partial_t - V_{IJ}\partial_x)\partial_x \phi^J = 0, \tag{5.16}$$

which resembles the equation of motion for a chiral boson theory. However, the terms K_{IJ} and V_{IJ} hinder this identification.

In this case, it is possible to diagonalize K_{IJ} and V_{IJ} to have a more tangible equation of motion for identification with the chiral boson theory. Since the matrix V_{IJ} is positive-definite, it is always possible to apply a transformation U to obtain the identity matrix $\mathbb{I}_{N\times N}$, i.e.,

$$V \to \mathbb{I} = UVU^T,$$

which corresponds to a basis change on the fields as $\phi \to U^T\phi$. Since the term containing the K-matrix also depends on the fields, the K-matrix will also feel the effect of this transformation as

$$K \to K_1 = UKU^T, \tag{5.17}$$

in which the sign of the eigenvalues of K is maintained since K and K_1 are congruent, but the absolute values of it may change. Then, use an orthogonal matrix transformation $O \in \mathcal{O}(N)$ to diagonalize K_1 as

$$K_1 \to (K_2)_{IJ} = OK_1O^T = \sigma_I|v_I|^{-1}\delta_{IJ}, \tag{5.18}$$

where $\sigma_I = \pm 1$, which corresponds to an orthogonal transformation in the fields $\phi \to O^T\phi$[1] [36].

Under this sequence of transformations, the Lagrangian becomes

$$L_{\text{edge}} = \frac{1}{4\pi}\sigma_I|v_I|^{-1}\delta_{IJ}\partial_t\phi^I\partial_x\phi^J - \frac{1}{4\pi}\delta_{IJ}\partial_x\phi^I\partial_x\phi^J, \tag{5.19}$$

and the corresponding equation of motion is given by

$$\begin{aligned}(\sigma_I|v_I|^{-1}\partial_t - \partial_x)\partial_x\phi^I &= 0\\ (\partial_t + \sigma_I|v_I|\partial_x)\partial_x\phi^I &= 0,\end{aligned} \tag{5.20}$$

which is the equation of motion for a chiral boson ϕ^I. The term $\sigma_I|v_I|$ corresponds to the velocity of the edge excitation captured by the field ϕ^I. Therefore, this equation shows that we have N gapless chiral (one-way propagation) boson fields ϕ^I in the edge theory. The edge density of the I-th condensate is

$$\rho^I = \frac{1}{2\pi}\partial_x\phi^I. \tag{5.21}$$

Notice that the direction of propagation of the edge modes σ_I came from the information encoded in the sign of the eigenvalues of the K-matrix. Therefore, it is possible to associate a signature (n_+, n_-) with the K-matrix where n_+ is related to the number of positive eigenvalues of K, which corresponds to excitations propagating in the right direction in the edge, and n_- is the number of negative eigenvalues of K, which corresponds to excitations propagating in the left direction in the edge. Since the sign of the velocity of each mode is related to the chirality of that mode, when there is an imbalance between the number of left and right propagating excitations, $n_+ \neq n_-$, the edge theory has a net chirality.

The concept of chirality is associated with the observable thermal Hall conductivity κ_H as

$$\kappa_H = (n_- - n_+)\frac{\pi^2 k_B^2}{3\hbar}T, \tag{5.22}$$

[1] Notice that the action of the transformation O under the identity matrix is trivial since $OO^T = \mathbb{I}$ and therefore the interaction term does not change under the orthogonal transformation

and thus if $n_+ \neq n_-$, the system has a non-zero quantized thermal Hall conductivity at the edge. On the other hand, when the number of excitations propagating to the right is the same as the excitations propagating to the left, $n_+ = n_-$, the thermal Hall conductivity vanishes at the edge and the system is non-chiral.

It has already been seen that the Wilson line in the x^1-direction corresponding to a quasiparticle $\mathbf{l}$ in the bulk is given by

$$W_l[x^0, x^2] = \exp\left(i\int l_I a_1^I dx^1\right). \tag{5.23}$$

Consider that the Wilson line is coming from the bulk at $-\infty$ to the edge at $x^1 = 0$. The corresponding equation for this Wilson line is

$$W_l[x^0, x^2] = \exp\left(i\int_{-\infty}^{0} l_I a_1^I dx^1\right). \tag{5.24}$$

Using the pure gauge configuration for a_1, one gets

$$W_l[x^0, x^2] = \exp\left(i\int_{-\infty}^{0} l_I \partial_1 \phi^I dx^1\right), \tag{5.25}$$

and assuming that $\phi^I \to 0$ when $x^1 \to -\infty$, we get

$$W_l[x^0, x^2] = \exp\left[i l_I \phi^I(x^0, x^2)\right] \tag{5.26}$$

$$\Psi_l = e^{il^T\phi}; \qquad l \in \mathbb{Z}^N, \tag{5.27}$$

where Ψ_l is an edge operator known as a vertex operator in the CFT language. To every quasiparticle $\mathbf{l}$ in the bulk, there is an associated vertex operator in the edge theory. This is the edge-bulk correspondence in disguise.

5.2 Perturbations

We aim to incorporate the vertex operator into the CS edge Lagrangian to analyze the stability of the edge modes under perturbations. To do that, the vertex operator must be local. As

$$\Psi_l = e^{il^T\phi}, \tag{5.28}$$

locality of the vertex operator requires that $l^T\phi$ to be local as well, since

$$\Psi_l(x)\Psi_l(y) = \exp\left(-[l^T\phi(x), l^T\phi(y)]\right)\Psi_l(y)\Psi_l(x), \tag{5.29}$$

in which the BCH theorem was used. Then, for the vertex operator at two different points to commute, the commutator between the fields $l^T\phi$ inserted at two different points x and y has to be an even integer, i.e.,

$$[l^T\phi(x), l^T\phi(y)] = \pi i l^T K^{-1} l \,\mathrm{sign}(x-y) \implies l^T K^{-1} l \in 2\mathbb{Z}, \tag{5.30}$$

where we used the Kac-Moody algebra to obtain this relation. To localize these fields at classical values, it is necessary to strengthen the above condition as

$$l^T K^{-1} l = 0, \tag{5.31}$$

which is Haldane's null vector condition [35].

Note that the statistics of the vertex operator Ψ_l is

$$\theta_l = e^{-i\pi l^T K^{-1} l} \tag{5.32}$$

If all edge modes move in the same direction (i.e., they are maximally chiral states), then all the eigenvalues of K are strictly positive [46]. This means that K^{-1} is positive-definite, i.e.,

$$\forall l \neq 0; \quad l^T K^{-1} > 0,$$

and, therefore, there is no vector charge $l \neq 0$ that satisfies the null vector condition. Thus, there is no local vertex operator in a maximally chiral phase, and then, no local perturbation can gap the edge modes. In this case, the edge modes will be robust against local perturbations and remain gapless, as the edge is protected.

When there is an imbalance between left and right edge modes ($n_+ \neq n_-$), the theory cannot be fully gapped. This is because a backscattering term or other local perturbations always gap out left and right edge modes in equal numbers. Suppose, for example, that $n_+ = 2$ and $n_- = 1$, that is, two right-mover modes and one left-mover mode at a particular edge. After diagonalizing the associated K-matrix, it is possible to separate this matrix into 2 sectors: one containing the non-chiral part of the K-matrix and the other containing the remaining positive eigenvalue

$$K = \left[\begin{array}{cc:c} \lambda_1 & & \\ & -\lambda_2 & \\ \hdashline & & \lambda_3 \end{array}\right], \tag{5.33}$$

with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}^+$. In this example, suppose that the pair of counter-propagating modes in the non-chiral sector can be gapped. Notice that the sector with λ_3 is positive-definite and therefore there is no vector charge $l \neq 0$ that satisfies the null vector condition. Thus, this mode will remain gapless.

Thus, to fully gap out an edge, the number of left and right propagating modes must be the same ($n_+ = n_-$), that is, the edge must be non-chiral. For that reason, the dimension N of the K-matrix must be even with

$$n_+ = n_- = \frac{N}{2}.$$

In this case, it is possible to add local hermitian terms in the edge action

$$\frac{\tilde{C}_l}{2}e^{il^T\phi} + \frac{\tilde{C}_l^*}{2}e^{-il^T\phi} \sim \cos(l^T\phi + \alpha_l), \tag{5.34}$$

where $\tilde{C}_l \equiv C_l e^{i\alpha_l}$ is the coupling constant. This cosine term is called the Higgs perturbation term.

Two Higgs terms $\cos(l_1^T\phi + \alpha_{l_1})$ and $\cos(l_2^T\phi + \alpha_{l_2})$ are independent if and only if [46]

$$l_1^T K^{-1} l_1 = l_2^T K^{-1} l_2 = l_1^T K^{-1} l_2 = 0,$$

which means that they are independent only and only if the fields $l_I\phi^I$ form a pair of commuting variables under the K-M algebra.

Consider a set of N_M mutually commuting fields where N_M is not necessarily equal to N

$$M = \{l_n^T\phi^n;\ s.t.\ l_n^T K^{-1} l_m = 0, \forall n, m = 1, 2, \ldots, N_M\}.$$

Because of the Heisenberg Principle, all these fields can be simultaneously pinned at classical values. Therefore, the edge excitations associated with these fields can be gapped. However, this is a necessary condition, but not a sufficient one. In the absence of symmetries, it is usually possible to fully gap out the edge theory. When the edge excitations can be fully gapped out in the absence of symmetries, the corresponding phase is topologically trivial. However, some symmetries can forbid some Higgs terms, and, therefore, they cannot be added to the edge theory.

In this work, we will analyze only the case where there is no symmetry in the Abelian topologically ordered state. It is possible to analyze the edge theory by looking at local quasiparticle labels, as done in the bulk case.

It was already seen that a bulk quasiparticle l is local to any other quasiparticle l' if

$$l \in \Gamma_e; \qquad \Gamma_e = \{K\Lambda;\ \Lambda \in \mathbb{Z}^N\},$$

where Γ_e is the electron lattice. For each local quasiparticle l in the bulk, there is a local excitation $\Psi_l = \Psi_{K\Lambda}$ at the edge. Thus, it is possible to say that all local edge excitations are composed of N independent local microscopic degrees of freedom

$$\Psi_I^\dagger = e^{iK_{IJ}\phi^J}, \tag{5.35}$$

which are called "electron" creation operators for the I-th edge mode. Remember that $l = K\Lambda$ may be bosonic or fermionic depending on the K-matrix. Then, Φ_I will be a boson or a fermion depending on the diagonal elements of the K-matrix.

It is possible to write a general product of electron operators as

$$\Psi_{K\Lambda} = e^{i\Lambda^T K\phi}, \tag{5.36}$$

which is still local. Therefore, in this notation, the perturbation term is given by

$$\frac{\tilde{C}_\Lambda}{2} e^{i\Lambda^T K\phi} + h.c. \sim \cos(\Lambda^T K\phi + \alpha_\Lambda). \tag{5.37}$$

Since $l = K\Lambda$, Haldane's null condition becomes

$$\Lambda^T K \Lambda = 0, \tag{5.38}$$

so that $\{\Lambda_i\}$ are composed of $N/2$ linearly independent vectors. In this case, the cosine terms can be pinned at classical minima values

$$\Lambda_i^T K\phi = 2\pi n_i; \quad n_i \in \mathbb{Z}. \tag{5.39}$$

The gapped boundary induced by backscattering cosine terms in 5.37 can be interpreted as a particle condensate.

Notice that any vector $l_a \in \Gamma_e$ can be decomposed as

$$l_{a,I} = c_a m_I,$$

where c_a is the minimal integer vector with no common factor in its components. Since $\Lambda_i^T K\phi$ obtains a classical value at the gapped edge, so does the vertex operator

$$\Psi_l = e^{il^T\phi} = e^{i\Lambda^T K\phi} = e^{ic_a m^T\phi}$$

as

$$\langle e^{im^T\phi} \rangle = e^{i2\pi n_i/c_a}, \tag{5.40}$$

which means that the quasiparticle m is condensed at the edge.

5.3 Lagrangian Subgroups

It is possible to define particle condensation more consistently through Lagrangian subgroups. The particles that are condensed at the edge form a subgroup $\mathscr{M} \subset \mathscr{A}$, where $\mathscr{A}$ is a finite set containing all the distinct quasiparticles of the system.

The Lagrangian subgroup $\mathscr{M}$ is an abelian group with group multiplication given by particle fusion. The properties of this group are

1. Every two particles on $\mathscr{M}$ are mutually bosonic

   $$e^{i\theta_{mm'}} = 1, \forall m, m' \in \mathscr{M},$$

   so that it is possible to condensate every particle in simultaneously.

2. For every quasiparticle with integer vector $l \in \mathscr{A}$, it either satisfies

- $l = \sum_i c_i m_i \in \mathscr{M}$.
- $l \notin \mathscr{M}$.

  In this case, l has non-trivial statistics with at least one condensed particle $m \in \mathscr{M}$

  $$\forall l \notin \mathscr{M},\ \exists m \in \mathscr{M}\ s.t.\ e^{i\theta_{lm}} \neq 1,$$

  meaning that all quasiparticles $l \notin \mathscr{M}$ are confined after condensation of particles in the Lagrangian subgroup.

3. $\mathscr{M}$ is closed under fusion rules

$$m \times m' = m'' \in \mathscr{M};\ \forall m, m' \in \mathscr{M}.$$

4. This condition only applies to bosonic states where the diagonal elements of K are all even: All particles in $\mathscr{M}$ are bosonic particles

$$e^{i\theta_m} = 1, \forall m \in \mathscr{M}.$$

In a physical sense, $\mathscr{M}$ describes the set of particles that can be annihilated at the edge. An important result is that it is possible to find a set of N/2 linearly independent vectors satisfying Haldane's null condition

$$\Lambda_i^T K \Lambda_j = 0,$$

if and only if there exists a Lagrangian subgroup $\mathscr{M} = \{m_i\}$ at the edge, with $\Lambda_i^T = c_i K^{-1} m_i^T$. A stronger argument is that the edge states of an Abelian Chern-Simons theory described by a K-matrix can support a gapped edge if and only if the theory has a vanishing thermal Hall conductance ($n_+ = n_-$) and it has at least one Lagrangian subgroup $\mathscr{M}$ defined in this theory [41].

Every state that does not satisfy the condition above has a protected edge. In other words, if the edge theory does not have a Lagrangian subgroup, no perturbation can open a gap at the edge, and the modes will remain gapless under any perturbation. Reference [44] shows that Lagrangian subgroups are capable of classifying edge states. If two gapped edges have different Lagrangian subgroups, they are topologically distinct.

It is possible to separate topological phases into three categories [54]:

1. Topological phases with null thermal Hall coefficient that support a gapped boundary;

2. Topological phases with null thermal Hall coefficient that do not support a gapped boundary;

3. Topological phases with non-zero thermal Hall coefficient that can never be fully gapped.

In the next chapters, we will analyze examples regarding type-I and type-II topologically ordered states.

5.4 Edge state with $\nu = 2/3$

This state can be understood as the particle-hole conjugate of the Laughlin state at filling $\nu = 1/3$, since $\nu = 1 - 1/3 = 2/3$. The K-matrix and the charge vector associated with this state are

$$K = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad t = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{5.41}$$

where, because of the form of the K-matrix, it is possible to infer that the edge is non-chiral with two counter-propagating edge modes. The filling fraction

$$\nu = t^T K^{-1} t$$

of this state is given by $\nu = 2/3$, as expected.

Since the diagonal entries of the K-matrix are odd integers, the theory is fermionic and has $2|\text{Det}(K)| = 6$ independent lines, where the Wilson operators are

$$\mathbb{I}, e^{-i\oint a} \tag{5.42}$$
$$\mathbb{I}, e^{-i\oint \tilde{a}}, e^{-i2\oint \tilde{a}}, e^{-3i\oint \tilde{a}}. \tag{5.43}$$

However, as $|\text{Det}(K)| = 3$, there are 3 distinct quasiparticles, which are defined modulo local particles $l \sim l + K\tau$; $\tau \in \mathbb{Z}^2$. Considering $\Gamma^T = (n, m)$ and an arbitrary quasiparticle vector $l^T = (l_1, l_2)$, one has

$$l \propto l + \begin{bmatrix} n \\ -3m \end{bmatrix}. \tag{5.44}$$

Therefore, the first entry of the quasiparticle vector is defined mod 1, and the second one is defined mod 3.

Therefore, we can write the set of all quasiparticles as

$$\mathscr{A} = \{(0,0)^T, (0,1)^T, (0,2)^T\}. \tag{5.45}$$

The charge associated with quasiparticles of the form $l = (0, l_2)$ is given by

$$q_l = t^T K^{-1} l = -\frac{1}{3} l_2 \tag{5.46}$$

and the charge of each quasiparticle is

$$q_{(0,0)} = 0 \tag{5.47}$$

$$q_{(0,1)} = -\frac{1}{3} \tag{5.48}$$

$$q_{(0,2)} = -\frac{2}{3}. \tag{5.49}$$

Thus, in terms of the charges, the set of distinct quasiparticles is

$$\mathscr{A} = \{0, e/3, 2e/3\}. \tag{5.50}$$

The mutual statistics between two arbitrary quasiparticles $l^T = (0, l_2)$ and $m^T = (0, m_2)$ of the set in $\mathscr{A}$ is

$$\theta_{lm} = -\frac{2\pi}{3} l_2 m_2, \tag{5.51}$$

and the self-statistics is

$$\theta_l = -\frac{\pi}{3} l_2^2. \tag{5.52}$$

Thus, the self-statistics of each quasiparticle is

$$\theta_{(0,0)} = 0 \tag{5.53}$$

$$\theta_{(0,1)} = -\frac{\pi}{3} \tag{5.54}$$

$$\theta_{(0,2)} = -\frac{4\pi}{3} \tag{5.55}$$

A natural question to ask is whether it is possible to open a gap in the edge theory. The first condition of having the same number of counter-propagating edge modes is satisfied, since the signature of the K-matrix is zero. The next step is to verify whether Haldane's null condition is satisfied. For that, let us try to find $\Lambda^T = (a, b)$ that satisfy the null condition

$$\Lambda^T K \Lambda = 0 \implies a^2 - 3b^2 = 0. \tag{5.56}$$

Note that there is no integer solution for a and b. Thus, there is no Lagrangian subgroup for the phase at filling $\nu = 2/3$. Therefore, the edge is robust against local perturbations, and the edge modes will remain gapless.

5.5 Edge state with $\nu = 8/9$

The state at filling $\nu = 8/9$ can be understood as the particle-hole conjugate of the Laughlin state at filling $\nu = 1/9$, since $\nu = 1 - 1/9 = 8/9$. The K-matrix and the charge vector associated with this state are

$$K = \begin{bmatrix} 1 & 0 \\ 0 & -9 \end{bmatrix} \qquad t = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{5.57}$$

where, because of the form of the K-matrix, it is possible to see that the edge is non-chiral with two counter-propagating edge modes.

The filling fraction is

$$\nu = t^T K^{-1} t = 8/9, \tag{5.58}$$

as expected. Notice that $\det(K) = 9$ and therefore we have 9 distinct quasiparticles modulo local particles

$$\begin{aligned} l &\sim l + K\tau; \qquad \tau^T = (m, n) \in \mathbb{Z}^2 \\ l &\sim l + \begin{bmatrix} n \\ -9m \end{bmatrix}. \end{aligned} \tag{5.59}$$

Using a general form $l^T = (l_1, l_2)$ for the quasiparticles, the equivalence relations modulo local particles are

$$l_1 \sim l_1 + n; \quad n \in \mathbb{Z} \tag{5.60}$$

$$l_2 \sim l_2 - 9m; \quad m \in \mathbb{Z}, \tag{5.61}$$

and then

$$l_1 \in \frac{\mathbb{Z}}{\mathbb{Z}} = \{0\} \tag{5.62}$$

$$l_2 \in \frac{\mathbb{Z}}{9\mathbb{Z}} = \mathbb{Z}_9. \tag{5.63}$$

Thus, the set of distinct quasiparticles for this state is given by

$$\mathscr{A} = \{(1,0), (1,1), (1,2), \ldots, (1,8)\}. \tag{5.64}$$

Let us calculate the charge of an arbitrary quasiparticle $l = (1, l_2)$ as

$$q_l = t^T K^{-1} l = 1 - \frac{l_2}{9}. \tag{5.65}$$

Therefore, the charge of each quasiparticle in $\mathscr{A}$ is

$$\begin{aligned} q_{(1,0)} &= 1 \\ q_{(1,1)} &= 1 - \frac{1}{9} = 8/9 \\ q_{(1,2)} &= 1 - \frac{2}{9} = 7/9 \\ q_{(1,3)} &= 1 - \frac{3}{9} = 6/9 \\ q_{(1,4)} &= 1 - \frac{4}{9} = 5/9 \\ q_{(1,5)} &= 1 - \frac{5}{9} = 4/9 \\ q_{(1,6)} &= 1 - \frac{6}{9} = 3/9 \\ q_{(1,7)} &= 1 - \frac{7}{9} = 2/9 \\ q_{(1,8)} &= 1 - \frac{8}{9} = 1/9. \end{aligned}$$

Then, in terms of the charges, the set of distinct quasiparticles is

$$\mathscr{A} = \{1/9, 2/9, \ldots, 8/9, 1\}. \tag{5.66}$$

The mutual statistics between two quasiparticles $l^T = (1, l_2)$ and $m^T = (1, m_2)$ is given by

$$\theta_{lm} = 2\pi l^T K^{-1} m = 2\pi \left(1 - \frac{l_2 m_2}{9}\right). \tag{5.67}$$

In addition, the self-statistics of a quasiparticle $l^T = (1, l_2)$ is

$$\delta_l = \pi l^T K^{-1} l = \pi \left(1 - \frac{l_2^2}{9}\right), \tag{5.68}$$

which is defined modulo 2π. The self-statistics of each distinct quasiparticle in the bulk is

$$\begin{aligned}
\delta_{(1,0)} &= \pi \left(1 - \frac{l_2^2}{9}\right) = \pi \\
\delta_{(1,1)} &= \pi \left(1 - \frac{1}{9}\right) = \frac{8\pi}{9} \\
\delta_{(1,2)} &= \pi \left(1 - \frac{4}{9}\right) = \frac{5\pi}{9} \\
\delta_{(1,3)} &= \pi \left(1 - \frac{9}{9}\right) = 0 \\
\delta_{(1,4)} &= \pi \left(1 - \frac{16}{9}\right) = \frac{-7\pi}{9} = \frac{11\pi}{9} \\
\delta_{(1,5)} &= \pi \left(1 - \frac{25}{9}\right) = \frac{-16\pi}{9} = \frac{2\pi}{9} \\
\delta_{(1,6)} &= \pi \left(1 - \frac{36}{9}\right) = -3\pi = \pi \\
\delta_{(1,7)} &= \pi \left(1 - \frac{49}{9}\right) = \frac{-40\pi}{9} = \frac{-14\pi}{9} \\
\delta_{(1,8)} &= \pi \left(1 - \frac{64}{9}\right) = \frac{-55\pi}{9} = \frac{17\pi}{9}
\end{aligned}$$

We can ask whether the edge theory related to the filling $\nu = 8/9$ can be gapped. The first condition of the same number of counter-propagating edge modes is satisfied. The second condition to be able to gap the edge is to find the null vector that satisfies

$$\Lambda^T K \Lambda = 0. \tag{5.69}$$

Note that $\Lambda^T = (3, \pm 1)$ are valid null vectors. The existence of a vector Λ that satisfies Haldane's null condition, along with the non-chirality of the edge, provides a sufficient criterion that the edge modes can be gapped.

Notice that the set $\mathscr{M} = \{1, 6/9, 3/9\} = \{(1,0), (1,3), (1,6)\}$ obeys the conditions 1, 2, and 3 to be a Lagrangian subgroup. Let us prove this statement as

1. $e^{i\theta_{mm'}} = 1, \forall m, m' \in \mathscr{M}$.

$$\theta_{03} = \theta_{06} = 2\pi \implies e^{i\theta_{03}} = e^{i\theta_{06}} = 1$$
$$\theta_{36} = 2\pi\left(1 - \frac{18}{9}\right) = -2\pi \implies e^{i\theta_{36}} = 1.$$

2. $\forall l \notin \mathscr{M}, \exists m \in \mathscr{M}$ $s.t.$ $e^{i\theta_{lm}} \neq 1$.

$$\begin{aligned}
\theta_{1m} &= 2\pi\left(1 - \frac{m}{9}\right) = 2\pi\left(\frac{9-m}{9}\right) \neq 0 \mod 2\pi \text{ for } m = 3 \text{ and } m = 6\\
\theta_{2m} &= 2\pi\left(\frac{9-2m}{9}\right) \neq 0 \mod 2\pi \text{ for } m = 3 \text{ and } m = 6\\
\theta_{4m} &= 2\pi\left(\frac{9-4m}{9}\right) \neq 0 \mod 2\pi \text{ for } m = 3 \text{ and } m = 6\\
\theta_{5m} &= 2\pi\left(\frac{9-5m}{9}\right) \neq 0 \mod 2\pi \text{ for } m = 3 \text{ and } m = 6\\
\theta_{7m} &= 2\pi\left(\frac{9-7m}{9}\right) \neq 0 \mod 2\pi \text{ for } m = 3 \text{ and } m = 6\\
\theta_{8m} &= 2\pi\left(\frac{9-2m}{9}\right) \neq 0 \mod 2\pi \text{ for } m = 3 \text{ and } m = 6.
\end{aligned}$$

3. $m \times m' = m'' \in \mathscr{M}; \forall m, m' \in \mathscr{M}$.

   Remember that m is defined mod 1 in the first entry and mod 9 in the second one. With $m \in \mathscr{M} = \{(1,0),(1,3),(1,6)\}$, one has

$$\begin{aligned}
&(1,3).(1,6) = (1, 18 \bmod 9) = (1,0) \in \mathscr{M}\\
&(1,0).(1,3) = (1,0) \in \mathscr{M}\\
&(1,0).(1,0) = (1,0) \in \mathscr{M}.
\end{aligned}$$

Thus, we proved that the elements of $\mathscr{M}$ form indeed a Lagrangian subgroup. As a result of the existence of one Lagrangian subgroup, consistently with the previous result, the edge is not protected, and cosine perturbation terms can gap the modes. Note that the element $(1,3) \in \mathscr{A}$ with charge $q = 6/9$ generates the whole Lagrangian subgroup.

Let us construct the edge theory for the $\nu = 8/9$ state with Lagrangian

$$L = \frac{1}{4\pi}K_{IJ}\partial_t\phi^I\partial_x\phi^J - \frac{1}{4\pi}V_{IJ}\partial_x\phi^I\partial_x\phi^J. \tag{5.70}$$

Suppose the K-matrix in equation 5.57 and

$$V = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}, \tag{5.71}$$

being the interaction matrix where v corresponds to a positive velocity.

Since K_{IJ} is diagonal, it is possible to consider the edge modes ϕ^1 and ϕ^2 as being decoupled. Then, the edge mode ϕ^1 is related to a single-component state with $\nu = 1$ and

$K = 1$, propagating to the right with positive-definite velocity v_1. The edge Lagrangian for the first system is given by

$$L_1 = \frac{1}{4\pi}\partial_t\phi^1\partial_x\phi^1 - \frac{v_1}{4\pi}(\partial_x\phi^1)^2. \tag{5.72}$$

On the other hand, the edge mode ϕ^2 is related to a single-component state with $\nu = 1/9$ with opposite chirality, that is, with $K = -9$, propagating to the left with positive-definite velocity v_2. The edge Lagrangian for this system is given by

$$L_2 = -\frac{9}{4\pi}\partial_t\phi^2\partial_x\phi^2 - \frac{v_2}{4\pi}(\partial_x\phi^2)^2. \tag{5.73}$$

It was already seen that the electron creation operator for a multi-component Chern-Simons edge theory is

$$\Psi_I^\dagger = e^{iK_{IJ}\phi^J}. \tag{5.74}$$

Therefore, the electron creation operator for the mode ϕ^1 and for the ϕ^2 mode in our topological state are

$$\Psi_1^\dagger = e^{i\phi^1} \qquad \Psi_2^\dagger = e^{-9i\phi^2}. \tag{5.75}$$

The cosine perturbation term in terms of the K-matrix and the null vector Λ is given by

$$\cos(\Lambda^T K\phi - \alpha), \tag{5.76}$$

which represents how local particles can scatter from the forward propagating mode to the backward one. Using $\Lambda^T = (3, \pm 1)$ and $\phi^T = (\phi^1, \phi^2)$, the argument in the cosine perturbation term is

$$\Lambda^T K\phi = 3\phi^1 \mp 9\phi^2. \tag{5.77}$$

Thus, two perturbation terms can open a gap in our system

$$C\cos(3\phi^1 \pm 9\phi^2) = C\cos[3(\phi^1 \pm 3\phi^2)], \tag{5.78}$$

for large coupling C. These cosine perturbations are called Higgs terms. The edge will be gapped once $\phi^1 + 3\phi^2$ or $\phi^1 - 3\phi^2$ are localized at classical values by the Higgs term. The terms $\phi^1 + 3\phi^2$ and $\phi^1 - 3\phi^2$ cannot be simultaneously localized due to the K-M algebra [46]

$$[\phi^I(x), \partial_y\phi^J(y)] = 2\pi i(K^{-1})^{IJ}\delta(x-y) \tag{5.79}$$

$$(K^{-1})^{IJ} = 0 \text{ to } I \neq J$$

$$[\partial_x\phi^1(x), \partial_y\phi^1(y)] = 2\pi i\partial_x\delta(x-y) \tag{5.80}$$

$$[\partial_x\phi^2(x), \partial_y\phi^2(y)] = -\frac{2\pi i}{9}\partial_x\delta(x-y) \tag{5.81}$$

$$[\partial_x\phi^1(x), \partial_y\phi^2(y)] = 0. \tag{5.82}$$

Let us analyze both perturbation terms separately and in detail.

5.5.1 Superconductor process

The state at filling $\nu = 8/9$ can be interpreted as being in proximity to a superconductor. The superconductivity term is related to the null vector $\Lambda^T = (3, 1)$ and the cosine term

$$\cos(3\phi^1 - 9\phi^2) \sim e^{i(3\phi^1 - 9\phi^2)} + h.c, \tag{5.83}$$

where for simplicity, the phase $\alpha = 0$ was fixed.

This system has the correspondence

$$e^{3i\phi^1} = (\Psi_1^\dagger)^3,$$

which corresponds to the creation of 3 electrons in mode ϕ^1 and

$$e^{-9i\phi^2} = \Psi_2^\dagger,$$

which means that 3 electrons in mode ϕ^1 are scatter to a hole on mode ϕ^2. Notice that the superconductivity term does not conserve charge. The h.c. term represents the inverse process.

Due to $\Lambda^T K\phi = c_i m_i^T \phi$, then $\Lambda^T K = (3, -9)$. Therefore, $m^T = (1, -3)$ is condensed at the gapped edge. The particle $m^T = (1, -3)$ corresponds to a particle with braiding exchange

$$\delta_m = \pi \begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/9 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \tag{5.84}$$

$$= 0, \tag{5.85}$$

which represents a bosonic particle. Notice that $m^T = (1, -3)$ is equivalent to $(1, 6)$, differing by the addition of a local particle. Therefore, $m^T = (1, -3)$ behaves as a boson, in contrast to $(1, 6)$ which have fermionic statistics. The particle condensed at the edge is the particle $(1, 6)$, belonging to the Lagrangian subgroup, with the insertion of a local particle. The other quasiparticles in the Lagrangian subgroup condense with m^T, since they have trivial braiding with $(1, 6)$. Additionally, the quasiparticles that are not in the Lagrangian subgroup at the edge become confined when the particles are condensed, since the braiding with the condensed particles is non-trivial, meaning that they cannot move freely anymore.

5.5.2 Backscattering process

A backscattering process can gap the edge of our theory at filling $\nu = 8/9$. This process is associated with the particular $\Lambda^T = (3, -1)$ and the cosine term is

$$\cos(3\phi^1 + 9\phi^2) \sim e^{i(3\phi^1 + 9\phi^2)} + h.c., \tag{5.86}$$

where the phase $\alpha = 0$ was fixed for simplicity. Notice that

$$e^{3i\phi^1} = (\Psi_1^\dagger)^3$$

corresponds to the creation of 3 electrons in mode ϕ^1 and

$$e^{9i\phi^2} = (\Psi_2^\dagger)^\dagger = \Psi_2$$

corresponds to the annihilation of 1 electron in mode ϕ^2. Thus, this backscattering term does not conserve charge as well. The h.c. term represents the inverse process.

Furthermore, since $\Lambda^T K\phi = c_i m_i^T \phi$, then $\Lambda^T K = (3, 9)$, which implies that $m^T = (1, 3)$ and $c_i = 3$. Therefore, the particle $m^T = (1, 3)$ is condensed at the gapped edge when we insert the backscattering perturbation. The particle $m^T = (1, 3)$ corresponds to a particle with braiding exchange

$$\begin{aligned} \delta_m &= \pi \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/9 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= 0, \end{aligned} \tag{5.87}$$

which represents a bosonic particle. This aligns with the statement that particles condensed at the edges have bosonic statistics. The other quasiparticles in the Lagrangian subgroup condense with m^T, since they have trivial braiding with $(1, 3)$. Additionally, the quasiparticles that are not in the Lagrangian subgroup at the edge become confined when the particles are condensed, since the braiding with the condensed particles is non-trivial, meaning that they cannot move freely anymore.

Let us analyze the backscattering term

$$\cos[3(\phi^1 + 3\phi^2)] \sim (\Psi_1^\dagger)^3(\Psi_2) + h.c.$$

for sufficiently large C. Consider a G-transformation in $GL(2, \mathbb{Z})$ acting on the fields as

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}, \tag{5.88}$$

that transforms ϕ^1 and ϕ^2 into φ and θ as

$$\begin{cases} \phi^2 = \varphi \\ \phi^1 + 3\phi^2 = \theta, \end{cases} \tag{5.89}$$

that does not change the compacity of the fields, since

$$e^{i\phi^2} = e^{i\varphi} \tag{5.90}$$

$$e^{i\phi^1} = e^{i(\theta - 3\varphi)}. \tag{5.91}$$

Thus, using $\theta = \phi^1 + 3\phi^2$, the cosine term becomes

$$\cos[3(\phi^1 + 3\phi^2)] = \cos(3\theta), \tag{5.92}$$

with $\theta = \phi^1 + 3\phi^2$. This interaction term breaks the $U(1) \times U(1)$ structure to $\mathbb{Z}_3 \times \mathbb{Z}_1$. The cosine term is pinned at the minima

$$\begin{aligned} 3\theta &= 2\pi n_i;\ n_i \in \mathbb{Z} \\ \theta &= \frac{2\pi n_i}{3}, \end{aligned} \tag{5.93}$$

with $n_i = 0, 1, 2$. Thus, the cosine term can be pinned at 3 different classical values.

Define the non-local operator $\sigma(x) = e^{i\Theta}$ identified with a $\mathbb{Z}_3$ clock variable

$$\sigma^3 = 1; \quad \sigma^\dagger = \sigma^2.$$

Each of the minima in euqation 5.93 defines a distinct ground-state, labeled by

$$\sigma(x)\, |\Phi_{n_i}\rangle = \omega^{n_i}\, |\Phi_{n_i}\rangle\,,$$

where $\omega = e^{2\pi i/3}$ and $|\Phi_{n_i}\rangle$ are the symmetry broken states since the system spontaneously breaks the $\mathbb{Z}_3$ symmetry by choosing a value for $\Theta(x)$. Note that, because of that, there are 3 degenerate ground-states [1].

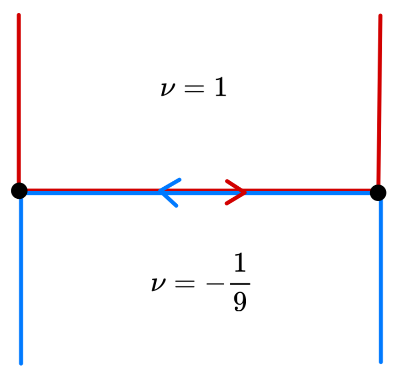


Figura 14 – Illustration of both phases $\nu = 1$ and $\nu = -1/9$, where the edges of both form an interface/branch cut separating both phases. The $\mathbb{Z}_3$ parafermions are bound to the endpoints of the branch cut (black dots in the figure).

The state at filling $\nu = 8/9$ can be interpreted as two different phases $\nu = 1$ and $\nu = -1/9$, where the edges of each phase form an interface with a branch cut that separates both phases, as shown in Figure 14. This system can host $\mathbb{Z}_3$ parafermions, which are anyons with non-Abelian statistics. The $\mathbb{Z}_3$ parafermions' zero modes are located at the endpoints of the gapped interface (black dots in Figure 14), where the defects are. Therefore, the $\mathbb{Z}_3$ parafermions are bound to these defects.

5.6 Edge states with $K = \mathbf{diag}(k_1, -k_2)$

More generally, let us study the gapability conditions for a general K-matrix

$$K = \begin{bmatrix} k_1 & 0 \\ 0 & -k_2 \end{bmatrix}; \qquad t = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{5.94}$$

In the previous examples, the cases $k_1 = 1;\ k_2 = -9$ and $k_1 = 1;\ k_2 = -3$ were studied, where in the first case it was possible to open a gap at the edge and in the second case the edge was robust against local perturbations. This example is based on the results of Ref. [1].

Type-1 topological phases (phases with null thermal Hall conductivity that support a gapped edge) arise when $k_1.k_2$ is a perfect square, since the null condition requires

$$\Lambda^T K \Lambda = 0 \implies a^2 k_1 = b^2 k_2, \tag{5.95}$$

and for a and b to have integer solutions, $k_1.k_2$ must be a perfect square. Notice that this is true by analyzing the previous examples, where $k_1.k_2 = 9$ in the first example, which is a perfect square, and it has a gapped edge. This is not true for the second example, where $k_1.k_2 = 3$, which is not a perfect square, so the topological phase is of type-2.

Then, it is possible to write any gappable interface with $K = \mathrm{diag}(k_1, -k_2)$ as

$$K = \begin{bmatrix} pn^2 & 0 \\ 0 & -pm^2 \end{bmatrix}; \qquad t = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{5.96}$$

with $p, m, n \in \mathbb{Z}^+$, $p = \gcd(k_1, k_2)$ and $\gcd(n^2, m^2) = 1$. The Kac-Moody algebra for the edge modes is

$$[\phi^I(x), \partial_y \phi^J(y)] = 2\pi i (K^{-1})^{IJ} \delta(x - y) \tag{5.97}$$

$$(K^{-1})^{IJ} = 0 \text{ to } I \neq J$$

$$[\partial_x \phi^1(x), \partial_y \phi^1(y)] = \frac{2\pi i}{pn^2} \partial_x \delta(x - y) \tag{5.98}$$

$$[\partial_x \phi^2(x), \partial_y \phi^2(y)] = -\frac{2\pi i}{pm^2} \partial_x \delta(x - y) \tag{5.99}$$

$$[\partial_x \phi^1(x), \partial_y \phi^2(y)] = 0. \tag{5.100}$$

Additionally, the local excitations are given by

$$\Psi_I^\dagger = e^{iK_{IJ}\phi^J},$$

and using

$$K\phi = \begin{bmatrix} pn^2 \phi^1 \\ -pm^2 \phi^2, \end{bmatrix}$$

the electron creation operators for modes ϕ^1 and ϕ^2 are

$$\Psi_1^\dagger = e^{ipn^2\phi^1} \tag{5.101}$$

$$\Psi_2^\dagger = e^{-ipm^2\phi^2}. \tag{5.102}$$

The perturbation term

$$C\cos(\Lambda^T K\phi)$$

is a local interaction that can open a gap at the edge if and only if there is at least one null vector $\Lambda^T = (a, b)$ satisfying

$$\Lambda^T K\Lambda = 0 \implies a^2n^2 = b^2m^2.$$

Notice that $\Lambda^T = (m, n)$; $a = m$, $b + n$ and $\Lambda^T = (m, -n)$; $a = m$, $b = -n$ satisfy the null condition above. The value of p does not interfere with the null vector Λ. Take $p = 1$ for simplicity. Then the K-matrix is

$$K = \begin{bmatrix} n^2 & 0 \\ 0 & -m^2 \end{bmatrix}, \tag{5.103}$$

and the electron creation operators are

$$\Psi_1^\dagger = e^{in^2\phi^1} \tag{5.104}$$

$$\Psi_2^\dagger = e^{-im^2\phi^2}. \tag{5.105}$$

As a result of the creation operators, the edge phase ϕ^1 has n^2 quasiparticles labeled by l_1^i with $i = 1, 2, \ldots, n^2$ and phase ϕ^2 has m^2 quasiparticles labeled by l_2^j with $j = 1, 2, \ldots, m^2$. The local recitations considering this label notation are given by

$$\Psi_1^\dagger = l_1^{k^1}$$

$$\Psi_2^\dagger = l_2^{k^2}.$$

The perturbation term with $\Lambda^T = (m, n)$ is

$$C\cos(\Lambda^T K\phi) = C\cos(mn^2\phi^1 - nm^2\phi^2)] = C\cos(mn\theta) \sim (\Psi_1^\dagger)^m(\Psi_2^\dagger)^n + h.c.,$$

with $\theta = n\phi^1 - m\phi^2$ and correspond to a superconducting term where m electrons in mode ϕ^1 are scatter to n holes on mode ϕ^2. On the other hand, the (backscattering) perturbation term with $\Lambda^T = (m, -n)$ is

$$C\cos(\Lambda^T K\phi) = C\cos(mn^2\phi^1 + nm^2\phi^2)] = C\cos(mn\theta) \sim (\Psi_1^\dagger)^m(\Psi_2)^n + h.c.,$$

with $\theta = n\phi^1 + m\phi^2$ which correspond to a backscattering term where m electrons are created in mode ϕ^1 and n electrons are annihilated in mode ϕ^2. This perturbation term is charge-conserving when $n = m$. The G-matrix transformation used to transform (ϕ^1, ϕ^2) to (φ, θ) was

$$G = \begin{bmatrix} r & s \\ n & \pm m \end{bmatrix}, \tag{5.106}$$

with the constraint $rm - sn = 1$ for $r, s \in \mathbb{Z}$. With that, we infer that the cosine term is pinned at the minima

$$\begin{aligned} nm\Theta &= 2\pi n_i;\ n_i \in \mathbb{Z} \\ \Theta &= \frac{2\pi n_i}{nm}, \end{aligned} \tag{5.107}$$

with $n_i = 0, 1, \ldots, nm$. Thus, the cosine perturbation term can be pinned at nm different classical values. This backscattering interaction term breaks the $U(1) \times U(1)$ structure to a $\mathbb{Z}_n \times \mathbb{Z}_m$ subgroup.

Define the non-local operator $\sigma(x) = e^{i\Theta(x)}$ with $\Theta = n\phi^1 + m\phi^2$ and identify it with a $\mathbb{Z}_{nm}$ clock variable

$$\sigma^{nm} = 1; \quad \sigma^\dagger = \sigma^{nm-1}.$$

Each of the minima in equation 5.107 defines a distinct ground-state, labeled by

$$\sigma(x) \left|\Phi_{n_i}\right\rangle = \omega^{n_i} \left|\Phi_{n_i}\right\rangle,$$

where $\omega = e^{2\pi i/nm}$ and $|\Phi_{n_i}\rangle$ are the symmetry broken states since the system spontaneously breaks the $\mathbb{Z}_{nm}$ symmetry by choosing a value for $\Theta(x)$. Note that, because of that, there are nm degenerate ground-states [1].

This system can be interpreted as two distinct phases A and B with $\nu = 1/n^2$ and $\nu = -1/m^2$, where the edges form an interface, as shown in Figure 15. This system can host $\mathbb{Z}_{nm}$ parafermions, which are located at the endpoints of the gapped interface (black dots in Figure 15). There is a branch cut separating the two phases, A and B, and the defects are at the endpoints of this branch cut. Thus, the $\mathbb{Z}_{nm}$ parafermions are bound to these defects [1].

Since we have the Kac-Moody algebra for the fields in equation 5.97, it is possible to gap the edge with either the backscattering term or with the superconducting term, but not with both terms simultaneously. Considering an allowed perturbation, it will have a particle condensation at the interface, where the particles m^T related to the null vectors Λ will be condensed at the edge. Using superconducting term with null vector $\Lambda^T = (m, n)$

$$\begin{aligned} \Lambda^T K\phi &= c_i m^T \phi \\ \begin{bmatrix} mn^2 & -nm^2 \end{bmatrix} &= c_i \begin{bmatrix} m_1 & m_2 \end{bmatrix} \\ mn\begin{bmatrix} n & -m \end{bmatrix} &= c_i \begin{bmatrix} m_1 & m_2 \end{bmatrix}, \end{aligned} \tag{5.108}$$

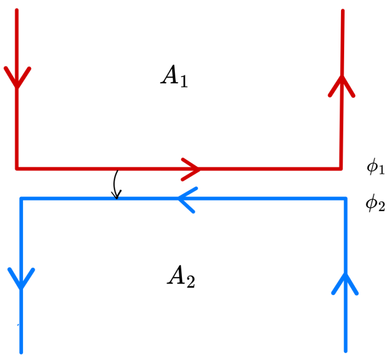


Figura 15 – Illustration of phases 1 and 2, where the edges of both form an interface. Edge mode excitation from phase 1 can tunnel to the edge of phase 2, where this process is illustrated in the curved arrow in the figure. The $\mathbb{Z}_{mn}$ parafermions are bound to the endpoints of the branch cut.

one has that $c_i = mn$, and $m^T = (n, -m)$ will be condensed at the gapped edge. Condensation of m^T implies the condensation of $e^{im^T\phi}$, since

$$\begin{aligned} \langle e^{im^T\phi} \rangle &= e^{i2\pi n_i/c_a} \\ \langle e^{i(n\phi^1 - m\phi^2)} \rangle &= e^{i2\pi n_i/nm}. \end{aligned} \tag{5.109}$$

The particle condensation at the edge also occurs for the backscattering term associated with $\Lambda^T = (m - n)$, but the particle condensing will be $m^T = (n, m)$.

Therefore, in summary

Backscattering	Superconductor
$\cos(mn^2\phi^1 + nm^2\phi^2)$	$\cos(mn^2\phi^1 - nm^2\phi^2)$
$\Lambda^T = (m, -n)$	$\Lambda^T = (m, n)$
$m^T = (n, m)$	$m^T = (n, -m)$

Tabela 1 – Summary of the results concerning the backscattering term and the superconductor term that are allowed to gap the edge of the state at filling $\nu = 8/9$.

6 CONCLUSION

In this dissertation, the bulk $U(1)$ Chern-Simons theory in 2+1 dimensions was reviewed along with the edge theory, when the boundary terms cannot be neglected. In this case, the edge theory contains a chiral boson field, which cancels the gauge anomaly originating from the bulk when a boundary is assumed.

The multi-component $U(1)^N$ Chern-Simons theory was also carefully analyzed using the K-matrix formulation. The equivalence between two theories with distinct K-matrices associated with a G-transformation, such that

$$K \to G^T K G,$$

where the fields a_μ^I and the vector charges t also transform under G, was presented. We showed an example in which a state with a filling fraction $\nu = 1/2n + 1$ in contact with an s-wave superconductor is equivalent to a state at filling $\nu = 1/8n + 4$.

In addition, within the context of the multi-component Chern-Simons theory, we presented the concept of anyonic symmetries, which are transformations that permute the anyons, preserving braiding statistics and fusion rules through a G-transformation in the quotient group $\text{Outer}(K)$. We showed an example of anyonic symmetry presented in the state at filling $\nu = 1/12$ realized by a $G_{11} = -1$ transformation that acts on the anyons as $m \to m^{11}$.

However, the analysis of the state at filling $\nu = 1/12$ signaled two additional anyonic symmetries besides the one realizable by G_{11}. To find the additional anyonic symmetries, we used the concept of stable equivalence, where the topological features of the bulk theory are preserved by adding a trivial sector to the K-matrix. For the theory with filling fraction $\nu = 1/12$, we added a trivial bosonic sector σ_x to the K-matrix to enlarge it. By performing this procedure, the other two G-matrices were found that realize the anyonic symmetries $m \to m^5$ and $m \to m^7$.

In the following, we review the edge theory of the $U(1)^N$ CS through the K-matrix formulation. The Lagrangian of the edge theory was obtained through bosonization in 1+1 dimensions, and the Wilson operators at the edge were formulated as the vertex operators. The conditions for the gappability of the edge states of the $U(1)^N$ Chern-Simons theories were presented based on a literature review. The first condition for gappability is a non-chiral edge, that is, the same number of counter-propagating edge modes. However, this is not a sufficient condition. In addition, local vertex operators were used to study perturbations at the non-chiral edge, yielding Haldane's null condition for gappability. An edge can be fully gapped when the edge is non-chiral and there are $N/2$ null vectors respecting Haldane's condition, where N is the dimension of the K-matrix.

A stronger argument for gappability is the existence of at least one Lagrangian subgroup in the edge theory. Different Lagrangian subgroups yield topologically distinct gapped edges. The gappability of the edge states at different fillings was analyzed. It was reviewed that the edge state with $\nu = 2/3$ is robust against perturbations, even though the edge is non-chiral because there is no null vector in the edge that respects Haldane's null condition.

The state with $\nu = 8/9$ was also reviewed, where there are two null vectors $\Lambda^T = (3, \pm 1)$ that respect Haldane's condition. The null vector $\Lambda^T = (3, 1)$ corresponds to a superconductor perturbation term, where the system is interpreted as being in contact with a superconductor. In this case, the edge is gapped by the superconductor perturbation, resulting in particle condensation at the edge. On the other hand, the null vector $\Lambda^T = (3, -1)$ corresponds to a backscattering perturbation term, where electrons from one mode are scattered to electrons in the opposite direction in the other mode. In this case, the edge is gapped by the backscattering process, also resulting in particle condensation at the edge. Owing to the Kac-Moody algebra, both perturbations cannot simultaneously gap the edge.

Additionally, the gappability of the state with $\nu = 8/9$ was also analyzed through the existence of a Lagrangian subgroup. In this state, we showed the existence of quasiparticles in the edge theory that respects the condition for forming a Lagrangian subgroup. Therefore, the existence of at least one Lagrangian subgroup indicates that it is possible to open a gap in the edge theory of the state with $\nu = 8/9$.

The state with $\nu = 8/9$ can be interpreted as having two different phases with $\nu = 1$ and $\nu = -1/9$. The edges of these phases form an interface with a branch cut that separates both phases. At the endpoints of the interface, we present the existence of $\mathbb{Z}_3$ parafermion modes bound to the defects, being careful with the compactness of the fields.

Furthermore, the general edge states with a K-matrix, given by $K = \text{diag}(k_1, -k_2)$ were reviewed. The edges of these states are non-chiral; however, this condition does not guarantee that the edge can be gapped. As reviewed in the literature, the edge can only be gapped if $k_1 k_2$ is a perfect square, and in this case, there will be two perturbation terms allowed: a superconductor term and a backscattering one. Due to the Kac-Moody algebra, they cannot simultaneously gap the edge.

The state with K-matrix $K = \text{diag}(k_1, -k_2) = (n^2, -m^2)$ can be interpreted as having two different phases A with $\nu = 1/n^2$ and B with $\nu = -1/m^2$. The edges of these phases form an interface with a branch cut that separates the two phases. At the endpoints of the interface, we presented that there are $\mathbb{Z}_{mn}$ parafermions bound to the defects, in agreement with Ref. [1].

REFERÊNCIAS

[1] SANTOS, L. H.; HUGHES, T. L. Parafermionic Wires at the Interface of Chiral Topological States. *Phys. Rev. Lett.*, American Physical Society, v. 118, p. 136801, Mar 2017. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevLett.118.136801>.

[2] GREITER, M.; WILCZEK, F. *Fractional Statistics*. Annual Reviews, 2024. 131–157 p. Disponível em: <https://www.annualreviews.org/content/journals/10.1146/annurev-conmatphys-040423-014045>.

[3] SIMON, S. H. *Topological Quantum*. [S.l.]: Oxford University Press, 2023. ISBN 978-0-19-888672-3.

[4] BIERI, S.; FROHLICH, J. Physical principles underlying the quantum Hall effect. *Comptes Rendus Physique*, v. 12, p. 332–346, 2011.

[5] KHAN, M. N.; TEO, J. C. Y.; HUGHES, T. L. Anyonic symmetries and topological defects in Abelian topological phases: An application to the *ADE* classification. *Phys. Rev. B*, American Physical Society, v. 90, p. 235149, Dec 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.90.235149>.

[6] LANDAU, L. The Theory of Phase Transitions. *Nature*, v. 138, n. 3498, p. 840–841, nov. 1936. ISSN 1476-4687. Disponível em: <https://doi.org/10.1038/138840a0>.

[7] WEN, X.-G. Topological Order: From Long-Range Entangled Quantum Matter to a Unified Origin of Light and Electrons. *International Scholarly Research Notices*, John Wiley Sons, Ltd, v. 2013, n. 1, p. 198710, jan. 2013. ISSN 2356-7872. Disponível em: <https://doi.org/10.1155/2013/198710>.

[8] HANSSON, T.; OGANESYAN, V.; SONDHI, S. Superconductors are topologically ordered. *Annals of Physics*, v. 313, n. 2, p. 497–538, 2004. ISSN 0003-4916. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0003491604001046>.

[9] WEN, X.-G. Choreographed entanglement dances: Topological states of quantum matter. *Science*, v. 363, n. 6429, p. eaal3099, 2019. Disponível em: <https://www.science.org/doi/abs/10.1126/science.aal3099>.

[10] MCGREEVY, J. *Generalized Symmetries in Condensed Matter*. Annual Reviews, 2023. 57–82 p. Disponível em: <https://www.annualreviews.org/content/journals/10.1146/annurev-conmatphys-040721-021029>.

[11] GOMES, P. R. S. An introduction to higher-form symmetries. *SciPost Phys. Lect. Notes*, SciPost, p. 74, 2023. Disponível em: <https://scipost.org/10.21468/SciPostPhysLectNotes.74>.

[12] STANESCU, T. D. *Introduction to Topological Quantum Matter & Quantum Computation*. Second edition. S.l.: CRC PRESS, 2024. ISBN 978-1-04-004191-8 978-1-04-004198-7 978-1-00-322604-8.

[13] AROVAS, D.; SCHRIEFFER, J. R.; WILCZEK, F. Fractional Statistics and the Quantum Hall Effect. *Phys. Rev. Lett.*, American Physical Society, v. 53, p. 722–723, Aug 1984. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevLett.53.722>.

[14] WILCZEK, F. (Ed.). *Fractional statistics and anyon superconductivity*. [S.l.: s.n.], 1990.

[15] KHARE, A. Fractional statistics and Chern-Simons field theory in (2+1)-dimensions. p. 702–729, 8 1999.

[16] JAIN, J. K. *Composite Fermions*. [S.l.]: Cambridge University Press, 2007.

[17] FRADKIN, E. *Field Theories of Condensed Matter Physics*. 2. ed. [S.l.]: Cambridge University Press, 2013.

[18] TONG, D. Lectures on the Quantum Hall Effect. In: . [S.l.: s.n.], 2016.

[19] FELDMAN, D. E.; HALPERIN, B. I. Fractional charge and fractional statistics in the quantum Hall effects. *Reports on Progress in Physics*, IOP Publishing, v. 84, n. 7, p. 076501, jun 2021. Disponível em: <https://dx.doi.org/10.1088/1361-6633/ac03aa>.

[20] AROUCA, R.; CAPPELLI, A.; HANSSON, T. H. Quantum Field Theory Anomalies in Condensed Matter Physics. *SciPost Phys. Lect. Notes*, SciPost, p. 62, 2022. Disponível em: <https://scipost.org/10.21468/SciPostPhysLectNotes.62>.

[21] SACHDEV, S. *Quantum Phases of Matter*. [S.l.]: Cambridge University Press, 2023.

[22] DUNNE, G. V. Aspects of Chern-Simons Theory. In: COMTET, A. et al. (Ed.). *Aspects topologiques de la physique en basse dimension. Topological aspects of low dimensional systems*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1999. p. 177–263. ISBN 978-3-540-46637-6.

[23] PACHOS, J. K. *Introduction to Topological Quantum Computation*. [S.l.]: Cambridge University Press, 2012.

[24] GAIOTTO, D. et al. Generalized Global Symmetries. *JHEP*, v. 02, p. 172, 2015.

[25] SEIBERG, N.; WITTEN, E. Gapped boundary phases of topological insulators via weak coupling. *Progress of Theoretical and Experimental Physics*, v. 2016, n. 12, p. 12C101, nov. 2016. ISSN 2050-3911. Disponível em: <https://doi.org/10.1093/ptep/ptw083>.

[26] WITTEN, E. Three lectures on topological phases of matter. *La Rivista del Nuovo Cimento*, v. 39, n. 7, p. 313–370, jul. 2016. ISSN 1826-9850. Disponível em: <https://doi.org/10.1393/ncr/i2016-10125-3>.

[27] FRöHLICH, J. Gauge invariance and anomalies in condensed matter physics. *Journal of Mathematical Physics*, v. 64, n. 3, p. 031903, mar. 2023. ISSN 0022-2488. Disponível em: <https://doi.org/10.1063/5.0135142>.

[28] LAUGHLIN, R. B. Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations. *Phys. Rev. Lett.*, American Physical Society, v. 50, p. 1395–1398, May 1983. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevLett.50.1395>.

[29] WEN, X. G.; ZEE, A. Classification of Abelian quantum Hall states and matrix formulation of topological fluids. *Phys. Rev. B*, American Physical Society, v. 46, p. 2290–2301, Jul 1992. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.46.2290>.

[30] KHAN, M. N. et al. Fermion parity flips and majorana bound states at twist defects in superconducting fractional topological phases. *Phys. Rev. B*, American Physical Society, v. 95, p. 205112, May 2017. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.95.205112>.

[31] LU, Y.-M.; FIDKOWSKI, L. Symmetry-induced anyon breeding in fractional quantum Hall states. *Phys. Rev. B*, American Physical Society, v. 89, p. 115321, Mar 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.89.115321>.

[32] TEO, J. C.; HUGHES, T. L.; FRADKIN, E. Theory of twist liquids: Gauging an anyonic symmetry. *Annals of Physics*, v. 360, p. 349–445, set. 2015. ISSN 0003-4916. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0003491615001967>.

[33] CANO, J. et al. Bulk-edge correspondence in (2 + 1)-dimensional Abelian topological phases. *Phys. Rev. B*, American Physical Society, v. 89, p. 115116, Mar 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.89.115116>.

[34] WANG, J.; WEN, X.-G. Nonperturbative regularization of (1 + 1)-dimensional anomaly-free chiral fermions and bosons: On the equivalence of anomaly matching conditions and boundary gapping rules. *Phys. Rev. B*, American Physical Society, v. 107, p. 014311, Jan 2023. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.107.014311>.

[35] HALDANE, F. D. M. Stability of Chiral Luttinger Liquids and Abelian Quantum Hall States. *Phys. Rev. Lett.*, American Physical Society, v. 74, p. 2090–2093, Mar 1995. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevLett.74.2090>.

[36] WEN, X.-G. Topological orders and edge excitations in fractional quantum Hall states. *Advances in Physics*, Taylor & Francis, v. 44, n. 5, p. 405–473, 1995. Disponível em: <https://doi.org/10.1080/00018739500101566>.

[37] KOU, S.-P.; LEVIN, M.; WEN, X.-G. Mutual Chern-Simons theory for Z_2 topological order. *Phys. Rev. B*, American Physical Society, v. 78, p. 155134, Oct 2008. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.78.155134>.

[38] LEVIN, M. et al. Exactly soluble models for fractional topological insulators in two and three dimensions. *Phys. Rev. B*, American Physical Society, v. 84, p. 235145, Dec 2011. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.84.235145>.

[39] LEVIN, M.; STERN, A. Classification and analysis of two-dimensional Abelian fractional topological insulators. *Phys. Rev. B*, American Physical Society, v. 86, p. 115131, Sep 2012. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.86.115131>.

[40] CHENG, M. Superconducting proximity effect on the edge of fractional topological insulators. *Phys. Rev. B*, American Physical Society, v. 86, p. 195126, Nov 2012. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.86.195126>.

[41] LEVIN, M. Protected Edge Modes without Symmetry. *Phys. Rev. X*, American Physical Society, v. 3, p. 021009, May 2013. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevX.3.021009>.

[42] BARKESHLI, M.; JIAN, C.-M.; QI, X.-L. Twist defects and projective non-Abelian braiding statistics. *Phys. Rev. B*, American Physical Society, v. 87, p. 045130, Jan 2013. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.87.045130>.

[43] BARKESHLI, M.; JIAN, C.-M.; QI, X.-L. Classification of topological defects in Abelian topological states. *Phys. Rev. B*, American Physical Society, v. 88, p. 241103, Dec 2013. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.88.241103>.

[44] BARKESHLI, M.; JIAN, C.-M.; QI, X.-L. Theory of defects in Abelian topological states. *Phys. Rev. B*, American Physical Society, v. 88, p. 235103, Dec 2013. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.88.235103>.

[45] WANG, C.; LEVIN, M. Weak symmetry breaking in two-dimensional topological insulators. *Phys. Rev. B*, American Physical Society, v. 88, p. 245136, Dec 2013. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.88.245136>.

[46] LU, Y.-M.; VISHWANATH, A. Theory and classification of interacting integer topological phases in two dimensions: A Chern-Simons approach. *Phys. Rev. B*, American Physical Society, v. 86, p. 125119, Sep 2012. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.86.125119>.

[47] LIN, C.-H.; LEVIN, M. Generalizations and limitations of string-net models. *Phys. Rev. B*, American Physical Society, v. 89, p. 195130, May 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.89.195130>.

[48] WANG, J. C.; WEN, X.-G. Boundary degeneracy of topological order. *Phys. Rev. B*, American Physical Society, v. 91, p. 125124, Mar 2015. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.91.125124>.

[49] LU, Y.-M.; VISHWANATH, A. Classification and properties of symmetry-enriched topological phases: Chern-Simons approach with applications to Z_2 spin liquids. *Phys. Rev. B*, American Physical Society, v. 93, p. 155121, Apr 2016. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.93.155121>.

[50] MOROZ, S. et al. Topological order, symmetry, and Hall response of two-dimensional spin-singlet superconductors. *Phys. Rev. B*, American Physical Society, v. 95, p. 014508, Jan 2017. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.95.014508>.

[51] HEINRICH, C.; LEVIN, M. Solvable models for neutral modes in fractional quantum Hall edges. *Phys. Rev. B*, American Physical Society, v. 95, p. 205129, May 2017. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.95.205129>.

[52] SANTOS, L. H. et al. Symmetry-protected topological interfaces and entanglement sequences. *Phys. Rev. B*, American Physical Society, v. 98, p. 075131, Aug 2018. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.98.075131>.

[53] BALRAM, A. C.; JAIN, J. K.; BARKESHLI, M. F_n superconductivity of composite bosons and the 7/3 fractional quantum Hall effect. *Phys. Rev. Res.*, American Physical Society, v. 2, p. 013349, Mar 2020. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevResearch.2.013349>.

[54] GANESHAN, S.; LEVIN, M. Ungappable edge theories with finite-dimensional Hilbert spaces. *Phys. Rev. B*, American Physical Society, v. 105, p. 155137, Apr 2022. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.105.155137>.

[55] JI, W.; TANTIVASADAKARN, N.; XU, C. Boundary states of three dimensional topological order and the deconfined quantum critical point. *SciPost Phys.*, SciPost, v. 15, p. 231, 2023. Disponível em: <https://scipost.org/10.21468/SciPostPhys.15.6.231>.

[56] LUO, Z.-X. Gapped boundaries of $(3+1)$-dimensional topological order. *Phys. Rev. B*, American Physical Society, v. 107, p. 125425, Mar 2023. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.107.125425>.

[57] YUTUSHUI, M.; PARK, J.; MIRLIN, A. D. Localization and conductance in fractional quantum Hall edges. *Phys. Rev. B*, American Physical Society, v. 110, p. 035402, Jul 2024. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.110.035402>.

[58] RYU, S.; ZHANG, S.-C. Interacting topological phases and modular invariance. *Phys. Rev. B*, American Physical Society, v. 85, p. 245132, Jun 2012. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.85.245132>.

[59] CHEN, X.; WEN, X.-G. Chiral symmetry on the edge of two-dimensional symmetry protected topological phases. *Phys. Rev. B*, American Physical Society, v. 86, p. 235135, Dec 2012. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.86.235135>.

[60] CHEN, X. et al. Symmetry protected topological orders and the group cohomology of their symmetry group. *Phys. Rev. B*, American Physical Society, v. 87, p. 155114, Apr 2013. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.87.155114>.

[61] SULE, O. M.; CHEN, X.; RYU, S. Symmetry-protected topological phases and orbifolds: Generalized Laughlin's argument. *Phys. Rev. B*, American Physical Society, v. 88, p. 075125, Aug 2013. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.88.075125>.

[62] HSIEH, C.-T. et al. Symmetry-protected topological phases, generalized Laughlin argument, and orientifolds. *Phys. Rev. B*, American Physical Society, v. 90, p. 165134, Oct 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.90.165134>.

[63] CHENG, M.; GU, Z.-C. Topological Response Theory of Abelian Symmetry-Protected Topological Phases in Two Dimensions. *Phys. Rev. Lett.*, American Physical Society, v. 112, p. 141602, Apr 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevLett.112.141602>.

[64] HSIEH, C.-T.; MORIMOTO, T.; RYU, S. CPT theorem and classification of topological insulators and superconductors. *Phys. Rev. B*, American Physical

Society, v. 90, p. 245111, Dec 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.90.245111>.

[65] GU, Z.-C.; LEVIN, M. Effect of interactions on two-dimensional fermionic symmetry-protected topological phases with Z_2 symmetry. *Phys. Rev. B*, American Physical Society, v. 89, p. 201113, May 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.89.201113>.

[66] CHEN, X.; LU, Y.-M.; VISHWANATH, A. Symmetry-protected topological phases from decorated domain walls. *Nature Communications*, v. 5, n. 1, p. 3507, mar. 2014. ISSN 2041-1723. Disponível em: <https://doi.org/10.1038/ncomms4507>.

[67] SANTOS, L. H.; WANG, J. Symmetry-protected many-body Aharonov-Bohm effect. *Phys. Rev. B*, American Physical Society, v. 89, p. 195122, May 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.89.195122>.

[68] YOSHIDA, T.; MORIMOTO, T.; FURUSAKI, A. Bosonic symmetry-protected topological phases with reflection symmetry. *Phys. Rev. B*, American Physical Society, v. 92, p. 245122, Dec 2015. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.92.245122>.

[69] SULLIVAN, J.; CHENG, M. Interacting edge states of fermionic symmetry-protected topological phases in two dimensions. *SciPost Phys.*, SciPost, v. 9, p. 016, 2020. Disponível em: <https://scipost.org/10.21468/SciPostPhys.9.2.016>.

[70] LICHTMAN, T. et al. Bulk anyons as edge symmetries: Boundary phase diagrams of topologically ordered states. *Phys. Rev. B*, American Physical Society, v. 104, p. 075141, Aug 2021. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.104.075141>.

[71] NING, S.-Q. et al. Edge theories of two-dimensional fermionic symmetry protected topological phases protected by unitary abelian symmetries. *Phys. Rev. B*, American Physical Society, v. 104, p. 075151, Aug 2021. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.104.075151>.

[72] MANJUNATH, N.; PREM, A.; LU, Y.-M. Rotational symmetry protected edge and corner states in Abelian topological phases. *Phys. Rev. B*, American Physical Society, v. 107, p. 195130, May 2023. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.107.195130>.

[73] HEINRICH, C.; LEVIN, M. Criteria for protected edge modes with F_2 symmetry. *Phys. Rev. B*, American Physical Society, v. 98, p. 035101, Jul 2018. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.98.035101>.

[74] SOHAL, R. et al. Entanglement entropy of generalized Moore-Read fractional quantum Hall state interfaces. *Phys. Rev. B*, American Physical Society, v. 102, p. 045102, Jul 2020. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.102.045102>.

[75] BAO, Y. et al. *Mixed-state topological order and the errorfield double formulation of decoherence-induced transitions*. 2023. Disponível em: <https://arxiv.org/abs/2301.05687>.

[76] HANSSON, T. H. et al. Quantum Hall physics: Hierarchies and conformal field theory techniques. *Rev. Mod. Phys.*, American Physical Society, v. 89, p. 025005, May 2017. Disponível em: <https://link.aps.org/doi/10.1103/RevModPhys.89.025005>.

[77] BURNELL, F. *Anyon Condensation and its Applications*. Annual Reviews, 2018. 307–327 p. Disponível em: <https://www.annualreviews.org/content/journals/10.1146/annurev-conmatphys-033117-054154>.

[78] YANG, G.; FELDMAN, D. E. Influence of device geometry on tunneling in the $\nu = \frac{5}{2}$ quantum Hall liquid. *Phys. Rev. B*, American Physical Society, v. 88, p. 085317, Aug 2013. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.88.085317>.

[79] LINDNER, N. H. et al. Fractionalizing Majorana Fermions: Non-Abelian Statistics on the Edges of Abelian Quantum Hall States. *Phys. Rev. X*, American Physical Society, v. 2, p. 041002, Oct 2012. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevX.2.041002>.

[80] VAEZI, A. Fractional topological superconductor with fractionalized Majorana fermions. *Phys. Rev. B*, American Physical Society, v. 87, p. 035132, Jan 2013. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.87.035132>.

[81] CLARKE, D. J.; ALICEA, J.; SHTENGEL, K. Exotic non-Abelian anyons from conventional fractional quantum Hall states. *Nature Communications*, v. 4, n. 1, p. 1348, jan. 2013. ISSN 2041-1723. Disponível em: <https://doi.org/10.1038/ncomms2340>.

[82] MONG, R. S. K. et al. Universal Topological Quantum Computation from a Superconductor-Abelian Quantum Hall Heterostructure. *Phys. Rev. X*, American Physical Society, v. 4, p. 011036, Mar 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevX.4.011036>.

[83] VAEZI, A. Z_3 generalization of the Kitaev's spin-1/2 model. *Phys. Rev. B*, American Physical Society, v. 90, p. 075106, Aug 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.90.075106>.

[84] KLINOVAJA, J.; YACOBY, A.; LOSS, D. Kramers pairs of Majorana fermions and parafermions in fractional topological insulators. *Phys. Rev. B*, American Physical Society, v. 90, p. 155447, Oct 2014. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.90.155447>.

[85] CANO, J. et al. Chirality-protected Majorana zero modes at the gapless edge of Abelian quantum Hall states. *Phys. Rev. B*, American Physical Society, v. 92, p. 195152, Nov 2015. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.92.195152>.

[86] SON, J. H.; ALICEA, J. Commuting-projector Hamiltonians for chiral topological phases built from parafermions. *Phys. Rev. B*, American Physical Society, v. 97, p. 245144, Jun 2018. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.97.245144>.

[87] SANTOS, L. H. Parafermions in hierarchical fractional quantum hall states. *Phys. Rev. Res.*, American Physical Society, v. 2, p. 013232, Feb 2020. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevResearch.2.013232>.

[88] NAYAK, C. et al. Non-Abelian anyons and topological quantum computation. *Rev. Mod. Phys.*, American Physical Society, v. 80, p. 1083–1159, Sep 2008. Disponível em: <https://link.aps.org/doi/10.1103/RevModPhys.80.1083>.

[89] RAO, S. Introduction to abelian and non-abelian anyons. In: ______. *Topology and Condensed Matter Physics*. Singapore: Springer Singapore, 2017. p. 399–437. ISBN 978-981-10-6841-6. Disponível em: <https://doi.org/10.1007/978-981-10-6841-6_16>.

[90] READ, N.; GREEN, D. Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect. *Phys. Rev. B*, American Physical Society, v. 61, p. 10267–10297, Apr 2000. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevB.61.10267>.

[91] DELMASTRO, D.; GOMIS, J. Symmetries of abelian Chern-Simons theories and arithmetic. *Journal of High Energy Physics*, v. 2021, n. 3, p. 6, mar. 2021. ISSN 1029-8479. Disponível em: <https://doi.org/10.1007/JHEP03(2021)006>.